Analysis on fractals: An introduction

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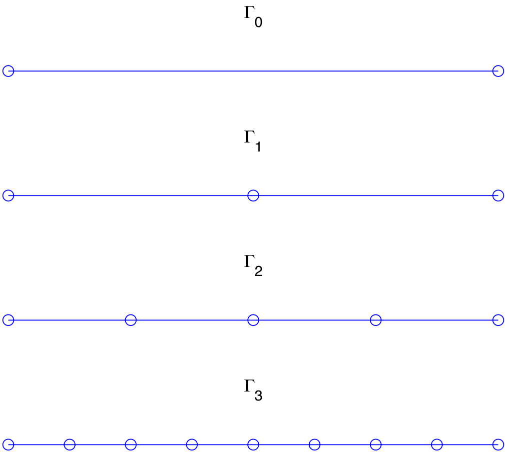
Outline

The unit interval as a self-similar fractal



The unit interval as a self-similar set

- Let $I = [0, 1]$, and $F_0(x) = \frac{1}{2}x$, $F_1(x) = \frac{1}{2}(x-1)+1$.
- $F_0 I = [0, 1/2]$, and $F_1 I = [1/2, 1]$. Thus

$$I = \bigcup_{i=0}^{1} F_i I.$$

- $\omega = (\omega_1, \omega_2, \ldots, \omega_m), \omega_i \in \{0, 1\} : |\omega| = m$.
- $F_\omega = F_{\omega_m} F_{\omega_{m-1}} \ldots F_{\omega_1}$.
-

$$I = \bigcup_{|\omega|=m} F_\omega I = \bigcup_{k=0}^{2^m-1} [\tfrac{k}{2^m}, \tfrac{k+1}{2^m}].$$

The unit interval as a limit of graphs

- I is the limit of graphs Γ_m with vertices in V_m defined as follows:

$$V_0 = \{0, 1\} \qquad V_m = \bigcup_{i=0}^{1} F_i V_{m-1} = \{k/2^m\}_{k=0}^{2^m}$$

- V_* is dense in I where

$$V_* = \bigcup_{m \geq 0} V_m = \{k/2^m : k = 0, 1, \ldots, 2^m\}_{m=0}^{\infty}.$$

An initial energy functional on the unit interval

- Let $u : \Gamma_0 \to \mathbb{R}$. Energy of u:

$$E_0(u) = (u(1) - u(0))^2.$$

- Find an extension $\tilde{u} : \Gamma_1 \to \mathbb{R}$ of u that minimizes the energy functional E_1:

$$\begin{aligned} E_1(\tilde{u}) &= (\tilde{u}(1) - \tilde{u}(1/2))^2 + (\tilde{u}(1/2) - \tilde{u}(0))^2 \\ &= (u(1) - \tilde{u}(1/2))^2 + (\tilde{u}(1/2) - u(0))^2 \end{aligned}$$

- Solution: $\tilde{u}(1/2) = \frac{u(1)+u(0)}{2} \implies E_1(\tilde{u}) = \frac{1}{2}E_0(u)$.
- More generally, given $u : \Gamma_m \to \mathbb{R}$ the minimum-energy extension $\tilde{u} : \Gamma_{m+1} \to \mathbb{R}$ satisfies $E_{m+1}(\tilde{u}) = \frac{1}{2}E_m(u)$.

The renormalized energy functional on the unit interval

Remark

It follows that $u : \Gamma_0 \to \mathbb{R}$ can be extended to $\tilde{u} : V_* \to \mathbb{R}$: $\tilde{u}_{|\Gamma_m}$ has minimum energy. Moreover, $E_m(\tilde{u}) = \frac{1}{2^m} E_0(u)$. Thus, given any function u on V_* the sequence $\{2^m E_m(u)\}_{m\geq 1}$ is a non decreasing and thus we have the following definition

Definition

Given $u : V_* \to \mathbb{R}$, the energy functional on u is defined by

$$\mathcal{E}(u) = \lim_{m\to\infty} 2^m E_m(u) \in [0, \infty]$$

A simplification of the (renormalized) energy functional

Remark

Observe that for $u \in \mathcal{C}^1(I)$, $\mathcal{E}(u) = \lim_{m\to\infty} 2^m E_m(u)$

$$\begin{aligned}
\mathcal{E}(u) &= \lim_{m\to\infty} \sum_{k=0}^{2^m-1} \tfrac{1}{2^m}\left(\frac{u(k2^{-m})-u((k+1)2^{-m})}{\frac{1}{2^m}}\right)^2 \\
&= \lim_{m\to\infty} \sum_{k=0}^{2^m-1} \tfrac{1}{2^m}(u'(c_k))^2 \\
&= \int_0^1 (u'(x))^2 dx
\end{aligned}$$

A weak definition of the Laplacian

Remark

Given $u, v \in \mathcal{C}^1(I)$ *with* $v(0) = v(1) = 0$ *we have*

$$\mathcal{E}(u, v) = \int_0^1 u'v'dx = -\int_0^1 u''vdx = -\langle \Delta u, v \rangle.$$

The Laplacian as a limit of graph Laplacians

Definition

Graph Laplacian: for $x \in V_* \setminus V_0$.

$$\Delta_m u(x) = \sum_{y \sim_m x} u(y) - 2u(x)$$
$$= u(x + 1/2^m) + u(x - 1/2^m) - 2u(x).$$

Moreover, For $u \in \mathcal{C}^1(I)$,

$$\Delta u(x) = \frac{d^2}{dx^2} u(x) = \lim_{m \to \infty} 4^m \Delta_m u(x).$$

Graph spectra and Fourier series

Observation

Recall that the solutions of $-\Delta u = \lambda u, \quad u(0) = u(1) = 0$ *are*

$$u_k(x) = \sin k\pi x \quad \textit{with} \quad \lambda_k = \pi^2 k^2,\ k = 1, 2, \ldots,$$

Given $m, k \geq 1$*, let* $u = u_{k_{|\Gamma_m}}, \lambda = \pi^2 k^2$*. It follows that*

$$-\Delta_m u = \lambda_m u \quad \textit{where} \quad \lambda_m = 4\sin^2\left(\frac{\sqrt{\lambda}}{2^{m+1}}\right).$$

Therefore, the restriction of any eigenfunction of $-\Delta$ *to any of the graphs* Γ_m *is an eigenfunction of* Δ_m*.*

Spectra of Δ_{m-1} and Δ_m

Lemma

$$\lambda_{m-1} = 4\sin^2(\tfrac{\sqrt{\lambda}}{2^m}) = \lambda_m(4-\lambda_m).$$

- *$u := u_{k_{|\Gamma_{m-1}}}$: $\Delta_{m-1}u = \lambda_{m-1}u$ can be extended to $\tilde{u}$: $\Delta_m\tilde{u} = \lambda_m\tilde{u}$.*
- *Consequently, the spectrum of Δ_m is completely determined by that of Δ_{m-1}. This is called the spectral decimation method.*

Observation

$$\lim_{m\to\infty} 4^m\lambda_m = \lim_{m\to\infty} 4^{m+1}\sin^2(\tfrac{\sqrt{\lambda}}{2^{m+1}}) = \lambda = \pi^2k^2$$

The spectral decimation method

Lemma (The spectral decimation method: Shima & Fukushima ($90s$); Rammal & Toulouse ($80s$))

Assume that

$$-\Delta_{m-1}u_{m-1}(x) = \lambda_{m-1}u_{m-1}(x) \quad x \in V_{m-1} \setminus V_0.$$

Let λ_m be such that $\lambda_{m-1} = \lambda_m(4-\lambda_m)$. Assume that $\lambda_m \neq 2$ and extend u_{m-1} to u_m by $u_m(x) = \frac{1}{2-\lambda_{m-1}}(u(y)+u(z))$ where y, z are the two neighbors of x in the graph Γ_m. Then

$$-\Delta_m u_m(x) = \lambda_m u_m(x) \quad x \in V_m \setminus V_0.$$

Recap

1. The unit interval is a limit of graphs.
2. $\Delta = \frac{d^2}{dx^2}$ is a limit of weighted graph Laplacians Δ_m.
3. The spectrum of $-\Delta$ is completely determined by the spectra of the graph Laplacians $-\Delta_m$.
4. The theory of Fourier series can be completely built from the spectral analysis of the graph Laplacians $-\Delta_m$

Question

Can this theory be extended to a non trivial setting, i.e., for a set other than the unit interval I?

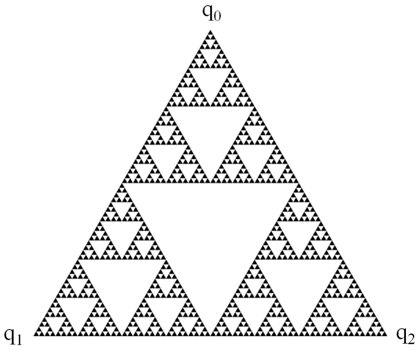
Definition

- What's a fractal?
- I know one when I see one!
- Here we consider self-similar fractals that are "barely" connected. These are known as *post critically finite (PCF) fractals*, a typical example of which is the Sierpinski gasket.

Definition

The fractal we consider below will be the unique non empty compact set K such that $K = \cup_{i=1}^{N} F_i(K)$ where $\{F_i\}_{i=1}^{N}$ are contraction maps. In addition $\#(F_i(K) \cap F_j(K)) < \infty$.

The Sierpinski Gasket (SG)



Construction of SG

Definition

Let $q_1 = (0,0)$, $q_0 = (1/2, \sqrt{3}/2)$, and $q_2 = (1,0)$. Define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_i(x) = \frac{1}{2}(x - q_i) + q_i$, $i = 0, 1, 2$.
The Sierpinski gasket is the unique nonempty compact subset SG of $\mathbb{R}^2$ such that

$$SG = \cup_{i=0}^{2} F_i(SG).$$

More generally, given $m \geq 1$,

$$SG = \cup_{|\omega|=m} F_\omega(SG).$$

SG as a limit of graphs

Observation

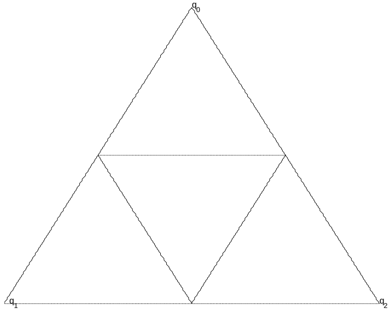
SG is the limit of graphs Γ_m with vertices in V_m where

$$V_0 = \{q_i\}_{i=0}^{2}, \qquad V_m = \bigcup_{i=0}^{2} F_i V_{m-1}.$$

V^* is dense in SG where

$$V_* = \bigcup_{m \geq 0} V_m$$

An initial energy functional on SG



An initial energy functional on SG

- Let $u : \Gamma_0 \to \mathbb{R}$. The energy of u is given by

$$E_0(u) = \sum_{k=0}^{2} (u(q_{k+1}) - u(q_k))^2.$$

- The extension $\tilde{u} : \Gamma_1 \to \mathbb{R}$ of u that minimizes E_1:

$$E_1(\tilde{u}) = \sum_{k,\ell=0}^{2} (\tilde{u}(F_k(q_{\ell+1})) - \tilde{u}(F_k(q_\ell)))^2$$

  is a *harmonic* function and

$$E_1(\tilde{u}) = \tfrac{3}{5} E_0(u).$$

- More generally, given $u : \Gamma_m \to \mathbb{R}$ the minimum-energy extension $\tilde{u} : \Gamma_{m+1} \to \mathbb{R}$ satisfies $E_{m+1}(\tilde{u}) = \frac{3}{5} E_m(u)$.

The renormalized energy functional on SG

Definition

Given $u : V_* \to \mathbb{R}$, the energy functional on u is defined by

$$\mathcal{E}(u) = \lim_{m\to\infty} \frac{5^m}{3^m} E_m(u) \in [0, \infty]$$

and

$$dom\mathcal{E} = \{u : V_* \to \mathbb{R} : \mathcal{E}(u) < \infty\}.$$

Remark

The Laplacian on SG can now be defined in a weak sense: Let $u \in dom\mathcal{E}$ and f be a continuous function on SG. Then $u \in dom\Delta$ with $\Delta u = f$ if $\mathcal{E}(u, v) = -\int_{SG} f v d\mu$ for all $v \in dom\mathcal{E}$ with $v_{|V_0} = 0$.

Pointwise definition of the Laplacian on SG

Definition

The graph Laplacian on Γ_m is defined by:

$$\Delta_m u(x) = \sum_{y \sim_m x} u(y) - 4u(x) \quad x \in V_m \setminus V_0.$$

Moreover, the Laplacian on SG is the operator defined by

$$\Delta u(x) = \tfrac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x).$$

The spectral decimation method on SG

Lemma (The spectral decimation method: Shima & Fukushima; Rammal & Toulouse)

Assume that
$-\Delta_{m-1}u_{m-1}(x) = \lambda_{m-1}u_{m-1}(x) \quad x \in V_{m-1} \setminus V_0.$
Let $\lambda_m : \lambda_{m-1} = \lambda_m(5-\lambda_m)$. *Assume that* $\lambda_m \neq 2, 5, 6$ *then* u_{m-1} *can be extended to a function* u_m *on* V_m *such that*

$$-\Delta_m u_m(x) = \lambda_m u_m(x) \quad x \in V_m \setminus V_0.$$

Conversely, if u_m *is an eigenfunction of* $-\Delta_m$ *with eigenvalue* λ_m, *then* $u_{m_{|V_{m-1}}}$ *is an eigenfunction of* $-\Delta_{m-1}$ *corresponding to the eigenvalue* λ_{m-1} *where* $\lambda_{m-1} = \lambda_m(5-\lambda_m)$.

Some properties of the spectrum of $-\Delta$ on SG

Observation

The solution of $-\Delta u = \lambda u$ on $SG \setminus V_0$ under Dirichlet or Neumann boundary conditions come from three families $\{\lambda_k^{(2)}, \lambda_k^{(5)}, \lambda_k^{(6)}\}_{k\geq 1}$ of eigenvalues of the graph Laplacian corresponding to the graph eigenvalues $2, 5, 6$.
$\lambda_k^{(2)}$ have all multiplicity 1, most of the $\lambda_k^{(5)}$, and $\lambda_k^{(6)}$ have very high multiplicities. Moreover, there exist localized eigenfunctions, i.e., eigenfunctions supported on very small subsets of SG, corresponding to the 5, and 6 series eigenvalues.

Fourier series on SG

Theorem

Let μ be the probability measure on SG that assigns the weight 3^{-m} to each set $F_\omega(SG)$ where $|\omega| = m$. Then there exists an orthonormal basis of eigenfunctions $\{u_k\}_{k=1}^\infty$ of $-\Delta$ on SG corresponding to eigenvalues $\{\lambda_k\}_{k=1}^\infty$. Consequently, and $u \in L^2(SG)$ has the following L^2 expansion

$$u = \sum_{k=1}^{\infty} \langle u, u_k \rangle u_k.$$

Some consequences of the existence of localized eigenfunctions

The existence of localized eigenfunctions of $-\Delta$ on SG has many consequences. In particular, using this

1. Heisenberg uncertainty principle type inequalities can be established on SG and other related fractals.
2. Spectra of Schrödinger type operators $H = -\Delta + V$ on SG.
3. Szegö-type limit theorems have been established on SG.
4. Convergence of Fourier series on SG.

Definition

Definition

For $j \geq 0$ a polynomial of degree less than or equal to j is any solution of

$$\Delta^{j+1} u = 0.$$

The set of all polynomials of degree less than or equal to j is denoted $\mathcal{H}^j$ and is a linear space of dimension $3(j+1)$. A basis for this space consists of $\{f_{ki} : 0 \leq k \leq j, i = 0, 1, 2\}$ where

$$\Delta^\ell f_{ki}(q_{i'}) = \delta_{\ell,k}\delta_{i,i'},$$

where $0 \leq \ell \leq j$, $i' = 0, 1, 2$.

Another basis of monomials for $\mathcal{H}^j$

Remark

Recall that on $I = [0,1]$, *and for* $j \geq 0$, *the monomial* $f_j(x) = \frac{x^j}{j!}$ *satisfies* $\frac{d^k f_j}{dx^k}(0) = 0$ *if* $0 \leq k < j$ *and* $\frac{d^j f_j}{dx^j}(0) = 1$. *On* SG *there are analogs of these monomials and they for a basis for* $\mathcal{H}^j$ *consisting of monomials* $\{P_{j,i} : i = 1,2,3; j \geq 0\}$.

A family of antisymmetric OP on SG

Definition

Fix $i = 1, 2$, or 3 and let $\{Q_j\}_{j=0}^{\infty}$ be the orthonormal system obtained by applying the Gram-Schmidt process to $\{P_{j,i}\}_{j\geq 0}$. Then $\{Q_j\}_{j=0}^{\infty}$ is a family of OP on SG, i.e.,

$$\langle Q_k, Q_\ell \rangle = \int_{SG} Q_k(x) Q_\ell(x) d\mu(x) = \delta_{k,\ell}.$$

Remark

Bases of polynomials on SG were used to solve numerically some differential equations (heat, wave, Schrödinger equations) on SG using finite element methods, K. Dalrymple, R. Strichartz and J. Vinson (1999).

A family of antisymmetric OP on SG

Theorem (R. S. Strichartz, E. K. Tuley, and K. A. O. (2013))

The OP polynomials $\{Q_k\}_{k\geq 0}$ defined above satisfies the following three-term recursion formula:

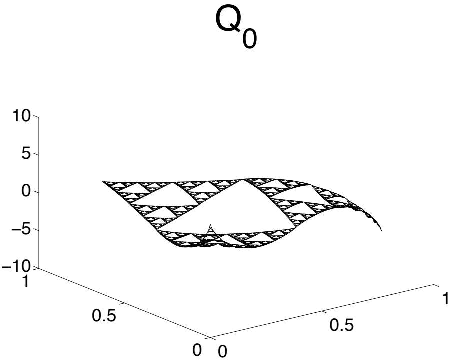
$$\sqrt{c_{k+1}}Q_{k+1}(x) = f_{k+1}(x) - b_k Q_k(x) - \sqrt{c_k}Q_{k-1}(x)$$

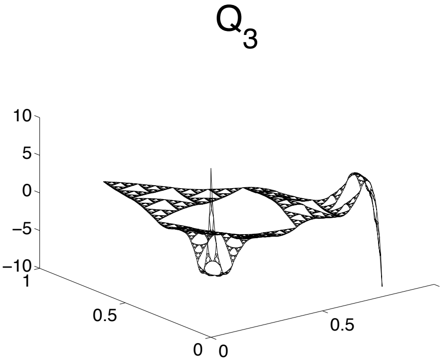
for some coefficients c_k, b_k and an auxilary polynomial $f_{k+1} \in \mathcal{H}^{k+1}$.
$Q_{-1}(x) = 0$, $Q_0(x) = d_0 P_{0,3}$.

Remark

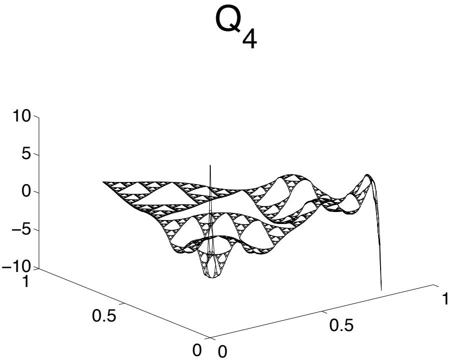
Analogues of Christoffel-Darboux formulas, Jacobi matrices associated to $\{Q_j\}_{j\geq 0}$

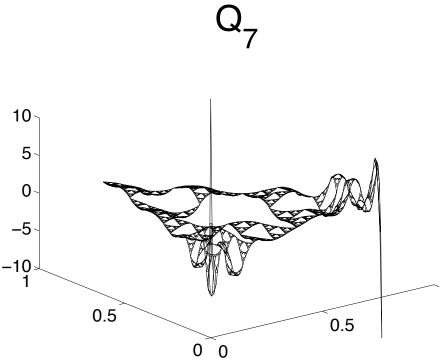
Examples of antisymmetric OP on SG



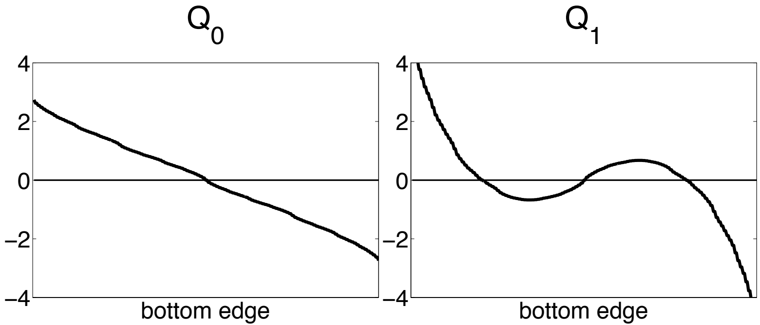


More examples

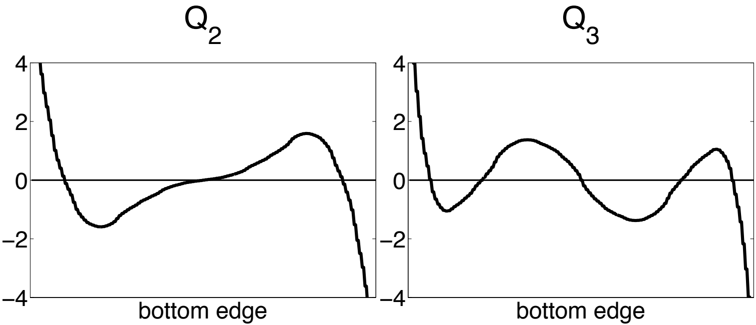




Restriction of the antisymmetric OP on the x axis



Restriction of the antisymmetric OP on the x axis



Heisenberg UP (Heisenberg-Weyl-Pauli, 1920's)

- Let $\|f\|_2 = \|\hat{f}\|_2 = 1$.
- Heisenberg Uncertainty Principle: [HUP]

$$\int |x - \mu_f|^2 \, |f(x)|^2 \, dx \int |\omega - \mu_{\hat{f}}|^2 \, |\hat{f}(\omega)|^2 \, d\omega \geq \frac{1}{16\pi^2}.$$

- A signal cannot be localized in position and momentum, or
- If you know where you are, you don't know where you are going; and if you know where you are going, you don't know where you are.

Heisenberg UP: Equivalent formulation

Note that

$$\int |\omega|^2 \, |\hat{f}(\omega)|^2 \, d\omega = \frac{1}{4\pi^2} \int |f'(x)|^2 \, dx := \frac{1}{4\pi^2} \mathcal{E}(f, f)$$

$$\int |x|^2 |f(x)|^2 dx = \frac{1}{2} \iint |x-y|^2 |f(x)|^2 |f(y)|^2 dx dy := \frac{1}{2} Var_d(|f|^2)$$

HUP can be written as

$$\iint |x-y|^2 |f(x)|^2 \, |f(y)|^2 \, dx \, dy \quad \int |f'(x)|^2 \, dx \geq \frac{1}{2}$$

equivalently,

$$Var_d(|f|^2) \mathcal{E}(f, f) \geq \frac{1}{2}.$$

Weak Uncertainty Principles

Theorem (K. A. Okoudjou, R. Strichartz, 2005)

There exists $C > 0$ *such that* $\forall u : \|u\|_2 = 1$, $\mathcal{E}_K(u,u) < \infty$ *and* $Var_d(|u|^2) \leq 1/2$, *then*

$$\mathcal{E}_K(u,u)\, Var_d(|u|^2) \geq C.$$

$\mathcal{E}_K(u,u)$ was defined earlier and

$$Var_d(|u|^2) = \iint_{SG\times SG} d(x,y)^{\gamma} |f(x)|^2 \, |f(y)|^2 \, d\mu(x)\, d\mu(y),$$

where d is a metric on SG given by

$$d(x,y) = \sup\{\tfrac{1}{\mathcal{E}(u,u)} : u \in \mathcal{C}(SG,\mathbb{R})\; u(x) = 0, u(y) = 1\}.$$

Weak UP on (K, d, μ)

Theorem (K. O/Saloff-Coste/Teplyaev, 2006)

Let (K, d, μ) be a metric measure space such that d is the effective resistance metric associated to the Dirichlet form $\mathcal{E}_K$. Under suitable conditions on μ and d, we prove that there exists $C > 0$ such that for all $u \in dom\mathcal{E}$, $\|u\|_2 = 1$,

$$Var_d(|u|^2)\,\mathcal{E}_K(u,u) \geq C.$$

Remarks

- If $K = \mathbb{R}$ we recover the HUP with a different proof.
- K is unbounded, and is not a graph.
- Similar results for bounded fractals, unbounded graphs.

Schrödinger operators on SG

- Schrödinger operators: $H = -\Delta + \chi$, where χ is a real-valued continuous function on SG.

- What is the asymptotics behavior of the spectrum of H?

- The potential χ "forces" high multiplicity eigenvalues of $-\Delta$ to split in cluster.

- The characteristic measure of this cluster converges to a measure.

Motivation: Schrödinger operators on the unit sphere

- Weinstein (1977): Consider $H = -\Delta + \chi$, Δ on a compact riemannian manifold, and χ is a smooth potential.
- Eigenvalues of H break into clusters.
- Example: on the unit n-sphere S^n, eigenvalues of $-\Delta$ are $\lambda_k = k(k+n-1), \quad k = 0, 1, 2, \ldots$. Each eigenvalue has multiplicity growing as a k^{n-1}.
- Guillemin (1978) has related results.

Spectra of Schrödinger operators on SG

Theorem (K. A. Okoudjou and R. S. Strichartz, 2007)

Let $H = -\Delta + \chi$ with χ continuous on SG, then the spectrum of H breaks into clusters which converges to a Borel measure.

The strong Szegö limit theorem on $[0, 2\pi)$

- Let

$$P_n : L^2([0, 2\pi)) \to span\{e^{im\theta} : 0 \leq m \leq n; 0 \leq \theta < 2\pi\}.$$

- If f is a function on $[0, 2\pi)$, $[f]$ denotes the multiplication operator by f:

$$[f]g = fg.$$

- $P_n[f]P_n$ is a Toeplitz matrix whose (k, l) entry is $\hat{f}_{k-l}$, the $(k - l)th$ Fourier coefficient of f.

The strong Szegö limit theorem on $[0, 2\pi)$

Theorem (G. Szegö, (1952))

If $f > 0$ and $f \in \mathcal{C}^{1+\alpha}$ where $\alpha > 0$, then the following holds:

$$\lim_{n\to\infty} \tfrac{1}{n+1} \log \det P_n[f]P_n = \tfrac{1}{2\pi} \int_0^{2\pi} \log f(\theta)\, d\theta.$$

Equivalently,

$$\lim_{n\to\infty} \tfrac{1}{n+1} \operatorname{Trace} \log P_n[f]P_n = \tfrac{1}{2\pi} \int_0^{2\pi} \log f(\theta)\, d\theta.$$

Szegö theorem on SG

Theorem (Strichartz, Rogers, K.O (2010); Ionescu, Rogers, K.O (2017))

The strong Szegö limit theorem holds on SG, and can be extended to pseudodifferential operators.

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Thank You!

http://www2.math.umd.edu/ okoudjou/