

Metrology Lab Hands-on Example

Calculating air density using the CIPM 2007 equation

7/30/2015

Goal: To introduce the CIPM 2007 equation for calculating the density of moist air.

Overview: Use Excel to calculate the various components used in the air density equation for a given temperature, pressure, and relative humidity.

Requirements: Environmental measuring equipment and a computer with Excel 2007 (or newer).

- 1) Density of moist air from NIST SOP 2 (NISTIR 6969)
 - a. Contribution due to water vapor (relative humidity) using the below equations.
 - i. What are the units associated with f ?

$$f = 1.000\ 62 + 3.14 \times 10^{-8}p + 5.6 \times 10^{-7}t^2$$

p = barometric pressure in Pascal t = temperature in °C

- ii. The following parameter has units of Pa.

$$p_{sv} = e^{AT^2+BT+C+D/T}$$

T = temperature in K $A = 1.237\ 884\ 7 \times 10^{-5}\ \text{K}^{-2}$ $B = -1.912\ 131\ 6 \times 10^{-2}\ \text{K}^{-1}$
 $C = 33.937\ 110\ 47$ $D = -6.343\ 164\ 5 \times 10^3\ \text{K}$

- iii. Finally, calculate water vapor contribution, x_v . What are its units?

$$x_v = (h/100)f \frac{p_{sv}}{p}$$

h = relative humidity in %

- b. Calculating the compressibility factor, Z . What are the units associated with Z ?

$$Z = 1 - \frac{p}{T} [a_0 + a_1 t + a_2 t^2 + (b_0 + b_1 t)x_v + (c_0 + c_1 t)x_v^2] + \frac{p^2}{T^2} (d + ex_v^2)$$

$a_0 = 1.581\ 23 \times 10^{-6}\ \text{K Pa}^{-1}$ $a_1 = -2.933\ 1 \times 10^{-8}\ \text{Pa}^{-1}$ $a_2 = 1.104\ 3 \times 10^{-10}\ \text{K}^{-1}\ \text{Pa}^{-1}$
 $b_0 = 5.707 \times 10^{-6}\ \text{K Pa}^{-1}$ $b_1 = -2.051 \times 10^{-8}\ \text{Pa}^{-1}$
 $c_0 = 1.989\ 8 \times 10^{-4}\ \text{K Pa}^{-1}$ $c_1 = -2.376 \times 10^{-6}\ \text{Pa}^{-1}$

$$d = 1.83 \times 10^{-11} \text{ K}^2 \text{ Pa}^{-2}$$

$$e = -0.765 \times 10^{-8} \text{ K}^2 \text{ Pa}^{-2}$$

- c. Calculate the air density, ρ_a , using the modified ideal gas equation, CIPM 2007, shown below. What are the units associated with ρ_a ?

$$\rho_a = \frac{pM_a}{ZRT} \left[1 - x_v \left(1 - \frac{M_v}{M_a} \right) \right]$$

$$M_a = 28.965 46 \times 10^{-3} \text{ kg/mol}$$

$$M_v = 18.015 281 7 \times 10^{-3} \text{ kg/mol}$$

$$R = 8.314 472 \text{ J K}^{-1} \text{ mol}^{-1}$$

2) Analysis of the results

- a. How does this compare to the results from the ideal gas equation below?

$$\rho_a = \frac{pM_a}{RT}$$

- b. How does this compare without the water-vapor term (note: there are many ways of calculating Z, often they only depend on p and T)?

$$\rho_a = \frac{pM_a}{ZRT}$$

- c. How do these densities compare with the density of normal air, $\rho_n = 0.0012 \text{ g/cm}^3$?
- d. What combination of T, p, and h would result in less dense air? More dense air?

Metrology Lab Hands-on Example

Effect of Balance Chamber Temperature

4/12/2017

Goal: To evaluate the use of in-chamber temperature measurement versus room temperature for mass calibrations involving air densities and the cubical coefficient of expansion. (This primarily affects echelon I calibrations because they utilize the CCE, whereas echelon II and III do not)

Overview: Balance chamber temperatures are often 0.1 °C to 0.7 °C higher than the room temperature, which means the air density experienced by the weight is likely smaller than that in the room. However, the increased temperature in the balance chamber also results in the weight expanding as it thermally equilibrates to the higher temperature (assuming it is initially outside the balance chamber).

Requirements: Pen, a few sheets of paper, and a calculator (or spreadsheet).

- 1) A 1 kg weight likely thermally equilibrates within 1 minute, given a 1 °C higher temperature in the balance chamber (see Benj for simulations). However, a 30 kg weight and a 0.5 °C temperature differential probably do not allow for significant equilibration during a 1 minute time span. This leads to the question, is it more important to use the higher temperature in MassCode to get the air density correct or does the CCE play a larger role and the room temperature should be used to get the volume of the weight correct?

Let us begin by calculating the volume, V_{20} , of a 30 kg weight at 20 °C ($\rho_s = 7.84 \text{ g/cm}^3$).

- 2) The first extreme assumes the weight does not thermally equilibrate at all, and remains at the room temperature. Using the below equation for thermal expansion, calculate the volume at the room temperature 19.7 °C and the associated mass of displaced air ($\rho_a = 1.1735 \text{ g/cm}^3$) given the CCE = 0.000045 /°C.

$$V_T = V_{20}[1 + \alpha(T - 20^\circ\text{C})]$$

- 3) Similar to (2), the other extreme assumes the weight completely equilibrates to the balance chamber temperature, 20.2 °C. Calculate the volume of the weight and the mass of displaced air ($\rho_a = 1.1714 \text{ g/cm}^3$).

- 4) Using the above two results, calculate the mass of displaced air for the ideal case (for our assumptions) in which the weight does not equilibrate to the balance chamber temperature, but is displacing the warmer air.

- 5) Which scenario (1 or 2) gets closer to the ideal answer in (4) and what does this tell you about the dominant factor when calculating the mass of displaced air?

- 6) For a balance with a division size of 1 mg, how much of an impact will the above results have?

- 7) What is the opposite ideal case (ie the weights remain in the balance chamber during the entire calibration)?

- 8) Finally, what about the worst case scenario in which the balance chamber temperature is not measured, so the room temperature is used. However, the weights fully equilibrate during their time in the balance chamber? Would adding a new uncertainty component to account for this be significant?

Metrology Lab Hands-on Example

Calculating Triangular Distribution Factor

11/16/2016

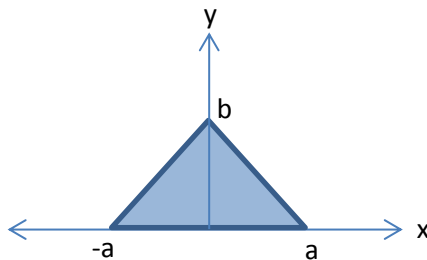
Goal: To introduce the distribution factors used when combining variances for different uncertainty components. This particular example involves the triangular distribution.

Overview: Using the provided picture, equations, and some calculus (see Aside 1 for examples) you will be able to calculate the triangular distribution factor.

Requirements: Pen and a few sheets of paper.

- 1) An introduction to distributions

All distributions need to be adjusted so they only contain 100 % of the items being plotted. In more mathematical verbiage, the distribution needs to be normalized. One way of thinking about this would be to cut the curve up into little strips and add all of those little rectangles up, and if the distribution is normalized we get 1. The reason for normalizing distributions is to provide equal footing for working with multiple distributions. Use the equations you derived in (1) to determine the area of the triangles in terms of a and b .



- 2) Now comes the normalization. Set the area equal to 1 and find b in terms of a .

3) Use the value for b to write two equations, one for the hypotenuse of each triangle, in terms of a , x , and y .

4) You are now ready to figure out the distribution factor, which is the standard deviation (σ) of the distribution. For a continuous distribution (as opposed to a discrete distribution, which looks like a histogram), the variance (σ^2) is defined by the following equation in which E is called the expectation value (more on this below):

$$\sigma^2 = E(x^2) - [E(x)]^2$$

The expectation value of a function (or product of functions) is the area under the curve. The notation above, taking $E(x^2)$ as an example, means the function of interest (in this case the equations for our two triangles) are multiplied by x^2 , and the area under the resulting equation is calculated. Similarly, $E(x)$ means the equations for our two triangles are multiplied by x and the area under the resulting equation is the expectation value.

- a. Beginning with the first term above, calculate $E(x^2)$. When finding the area (in the language of calculus, find the definite integral from $-\infty$ to ∞), you need to break the problem into 4 parts ($-\infty$ to $-a$, $-a$ to 0 , 0 to a , and a to ∞).

b. Now tackling the second term above, calculate $E(x)$. Similar to part (a), split the integral into 4 parts.

c. Combine the results of (a) and (b) to get σ^2 and σ .

Aside 1: A small introduction to integrals

0. What is an integral?

An integral is a method for calculating the area underneath a curve. Namely, we have an equation (e.g. $y = \dots$) involving a variable, often denoted as x , and we want to know the area between the x axis and the line drawn for y .

An integral is represented by the S-shaped symbol, \int . Sometimes there is a subscript and a superscript value, $\int_{-0.1}^{2.89}$, which means we want to know the area under the curve from $x = -0.1$ to $x = 2.89$ (this is a definite integral in math speak). If these limits are not present, the integral is called indefinite.

When y has a bunch of terms added or subtracted together, the integral of y can be separated into integrals for each of those values. (this is not true if terms are multiplied or divided)

$$\int a + b - cx - dhdx = \int adx + \int bdx + \int (-cx)dx + \int (-hx^2)dx$$

The above equation has many dx terms, this is mathematical notation to remind you which variable is being integrated. h is used in place of d in the 4th term so it would not be confused with all of the dx terms. Additionally, anything that does not have the integration variable in it, dx in this case, behaves as a constant and can be moved outside of the integral sign. Then our above example becomes,

$$\int a + b - cx - hx^2 dx = a \int 1dx + b \int 1dx - c \int xdx - h \int x^2 dx$$

For equations involving constants and x , x^2 , x^3 , etc. the following formula lets you calculate the integral:

- Increase the power of x by 1 (i.e. x becomes x^2 , x^2 becomes x^3 , etc.), and constants can be thought of as x^0 (so they become x^1).
- Divide the result by the new exponent (from the above parenthetical, $(x^2)/2$, $(x^3)/3$, and $(x^1)/1$)

Returning to our example equation and applying the two rules we learned above,

$$\int a + b - cx - hx^2 dx = ax + bx - \frac{cx^2}{2} - \frac{hx^3}{3} + C$$

The next term at the very right, C , is a constant of integration because we did not choose definite limits. An example of a definite integral is shown below.

3. Example 3: Calculate the integral of $y = 6x^2 + 5x + 1$ using limits of 1 and 3. There is no simple geometric comparison for this integral, which shows exactly why knowing how to calculate integrals can be useful.

4. Bonus example: The integral of x^{-1} is $\ln(x)$. Calculate the integral of $y = 3x + 2x^{-1} - 6x^{-2}$ using limits of 0 and 4.

Metrology Lab Hands-on Example

Effect of Sensitivity Weight on Drift

11/16/2016

Goal: To evaluate the use and choice of sensitivity weights in mass calibration procedures involving at least one double substitution with respect to drift (or other factors involving the balance response).

Overview: Choice of sensitivity weight is often counter intuitive (smaller is not always better, but neither is larger). The drift encountered during a double substitution depends greatly upon the nominal mass of the sensitivity weight.

Requirements: Pen, a few sheets of paper, and a calculator (or spreadsheet).

- 1) A bit regarding the purpose of sensitivity weights

The sensitivity weight measures the linearity of the balance in the given range of use (i.e. adding 1 mg to the balance pan results in a 1 mg increase on the display). This linearity is not guaranteed because balances often behave similar to springs, which are quadratic and not linear. Identify the portion of the below equation for SOP 4 (double substitution) evaluating the linearity of the balance response.

$$CM_x = CM_s + \left[\frac{(O_2 - O_1) + (O_3 - O_4)}{2} \right] \left[\frac{CM_{sw}}{O_3 - O_2} \right] + N_s - N_x$$

- 2) What is the ideal value for the above term you identified and what are its units?

- 3) Now to use some numbers and evaluate this in practice. Let us assume we have three identical balances, but one is brand new (row 1), a few years old (row 2), and a beat up, old balance (row 3). Fill in the below chart, which presumes different amounts of linear drift. $CM_{sw} = 50.08$ mg.

Drift (mg)	CM_{sw} (mg)	O_2 (mg)	O_3 (mg)	$O_3 - O_2$ (mg)	Drift = $O_3 - O_2 - CM_{sw}$ (mg)	$CM_{sw}/(O_3 - O_2)$
0.01	50.08	28.17	78.26			
0.03	50.08	28.17	78.28			
0.10	50.08	28.17	78.35			

- 4) Assuming these three balance behave linearly for a 200 mg sensitivity weight, $CM_{sw} = 199.96$ mg, fill out the analogous table below.

Drift (mg)	CM_{sw} (mg)	O_2 (mg)	O_3 (mg)	$O_3 - O_2$ (mg)	Drift = $O_3 - O_2 - CM_{sw}$ (mg)	$CM_{sw}/(O_3 - O_2)$
0.02	199.96	28.17	228.15			
0.04	199.96	28.17	228.17			
0.14	199.96	28.17	228.27			

- 5) Which sensitivity weight minimizes the effect of drift and why?

- 6) Now assume a 500 mg sensitivity weight, $CM_{sw} = 500.11$ mg, pushes these balances out of their linear range. Complete the table below.

Drift (mg)	CM_{sw} (mg)	O_2 (mg)	O_3 (mg)	$O_3 - O_2$ (mg)	Drift = $O_3 - O_2 - CM_{sw}$ (mg)	$CM_{sw}/(O_3 - O_2)$
0.31	500.11	28.17	528.59			
0.37	500.11	28.17	528.65			
0.54	500.11	28.17	528.82			

- 7) How do the results from the 500 mg sensitivity weight compare with the previous two?

- 8) Which sensitivity weight would you choose for each balance and why?

Metrology Lab Hands-on Example

Single and Double Substitution Weighing Designs

7/30/2015

Goal: To introduce two similar weighing designs and evaluate the resulting mass values and their standard deviations.

Overview: Choose two weights of the same nominal mass and compare them using an appropriate balance. DO NOT use a sensitivity weight (this will be a separate exercise). Be sure to record the temperature, pressure, and relative humidity during all calibrations (for calculating the air buoyancy corrections in a future exercise).

Requirements: Environmental measuring equipment, two weights of identical nominal mass, balance, conventional mass corrections of both weights.

- 1) Single substitution (SOP 7) from NISTIR 6969
 - a. After warming up the balance, make measurements by putting the standard on the balance, zeroing the balance, recording observation 1 (which will likely be 0.0), removing the standard, putting the unknown on the balance, recording observation 2, and removing the unknown. Repeat this process 10-12 times.
 - b. Calculate the conventional mass (CM_x) for each measurement using the following equation (N_s = nominal mass of the standard, N_x = nominal mass of the unknown).
$$CM_x = CM_s + O_2 - O_1 + N_s - N_x$$
 - c. Calculate the average value of the unknown's conventional mass.
 - d. Calculate the standard deviation of the unknown's convention mass values.

- 2) Double substitution (SOP 4) from NISTIR 6969
 - a. Make measurements by putting the standard on the balance, zeroing the balance, recording observation 1 (which will likely be 0.0), removing the standard, putting the unknown on the balance, recording observation 2, waiting the same amount of time it would take to remove the weight and put a new one on, recording observation 3, removing the unknown, placing the standard back on the balance, and recording observation 4. Repeat this process 10-12 times.
 - b. Calculate how 100 mg would appear on the balance (if 100 mg is too large or too small, use a more appropriate value), and add this value to all O_3 and O_4 (this is a fictitious sensitivity weight, CM_{sw}).
 - c. Calculate the conventional mass using the below formula and the CM_{sw} from (b).
$$CM_x = CM_s + \left[\frac{(O_2 - O_1) + (O_3 - O_4)}{2} \right] \left[\frac{CM_{sw}}{O_3 - O_2} \right] + N_s - N_x$$
 - d. Calculate the average value of the unknown's conventional mass.
 - e. Calculate the standard deviation of the unknown's convention mass values.

3) Analysis of results

- a. How do the average values of CM_x compare between the two methods?
- b. How do the standard deviations compare between the two methods?
- c. Using an F test at the 95 % confidence level, can the standard deviations be distinguished?
- d. If you were to compare these two methods again, how many runs would you make using each method?

Metrology Lab Hands-on Example

Single and Double Substitution Weighing Designs

(with sensitivity weights)

2015-10-30

Goal: To introduce two similar weighing designs and evaluate the resulting mass values and their standard deviations.

Overview: Choose two weights of the same nominal mass and compare them using an appropriate balance. USE a sensitivity weight (SW) this time. Be sure to record the temperature, pressure, and relative humidity during all calibrations (for calculating the air buoyancy corrections in a future exercise).

Requirements: Environmental measuring equipment, two weights of identical nominal mass, balance, conventional mass corrections of both weights. Choose appropriate (see NIST SOP 34) sensitivity weights and record their true masses.

1) Single substitution (SOP 7) from NISTIR 6969

- After warming up the balance, make measurements by putting the standard on the balance, zeroing the balance, recording observation 1 (which will likely be 0.0), adding the SW, recording observation 2 and checking to make sure the reading is acceptable (< 2 % of the true mass), removing the standard and SW, putting the unknown on the balance, recording observation 3, and removing the unknown. Repeat this process 10-12 times.
- Calculate the conventional mass (CM_x) for each measurement using the following equation (N_s = nominal mass of the standard, N_x = nominal mass of the unknown).
$$CM_x = CM_s + O_3 - O_1 + N_s - N_x$$
- Calculate the average value of the unknown's conventional mass.
- Calculate the standard deviation of the unknown's convention mass values.

2) Double substitution (SOP 4) from NISTIR 6969

- Make measurements by putting the standard on the balance, zeroing the balance, recording observation 1 (which will likely be 0.0), removing the standard and putting the unknown on the balance, recording observation 2, adding the SW, recording observation 3, removing the unknown and SW, putting the standard and SW on the balance, and recording observation 4. Repeat this process 10-12 times.
- Calculate the conventional mass using the below formula and the CM_{sw} from (b).
$$CM_x = CM_s + \left[\frac{(O_2 - O_1) + (O_3 - O_4)}{2} \right] \left[\frac{CM_{sw}}{O_3 - O_2} \right] + N_s - N_x$$
- Calculate the average value of the unknown's conventional mass.
- Calculate the standard deviation of the unknown's convention mass values.

3) Analysis of results

- a. How do the average values of CM_x compare between the two methods?
- b. How do the standard deviations compare between the two methods?
- c. Using an F test at the 95 % confidence level, can the standard deviations be distinguished?
- d. If you were to compare these two methods again, how many runs would you make using each method?
- e. How do the results from these procedures compare with the ones performed without sensitivity weights?

Sample questions similar to those on NIST Fundamentals of Metrology test

Name: _____

Date: _____

Instructions: Show your work, be careful regarding units, do not round your answers, and circle all of your final answers.

Metric unit prefixes (eg 5 mm = 5×10^{-3} m): $\mu = 1 \times 10^{-6}$, m = 1×10^{-3} , c = 1×10^{-2} , k = 1×10^3

Conversion factors: 1 in = 2.54 cm, 1 lb = 0.45359237 kg

1) Perform the following unit conversions.

a. 62 mg to ___kg

b. -0.42 μ g to ___mg

c. 0.348 kg to ___lb

d. 34.2902 lb to ___g

e. 0.02093 m to ___cm

f. 1.2834 in to ___cm

g. 0.062389 yards to ___cm

h. 1.0 m^3 to ___ cm^3

i. 22 in^3 to ___ cm^3

- 2) Danny the intern decides to use a combination of avoirdupois and metric tare weights when calibrating a 100 kg weight. The two tare weights are 500 g (density = 8030 kg/m^3) and 1.3426 lb (density = 0.28 lb/in^3).

Calculate the volume of each tare weight in cm^3 .

- 3) Using the above information, calculate the effective density of the two tare weights in g/cm^3 .

- 4) Air buoyancy corrections for high-precision calibrations require the density of air in the room during the time of calibrations. A modified form of the ideal gas law is used to calculate the air density based upon the temperature, pressure, and relative humidity.

$$e_s = (1.3146 \times 10^9) e^{\left(\frac{-5315.56}{t+273.15}\right)}$$

Using a temperature, t , of 22.18 °C, calculate the pressure correction factor, e_s , using the equation given above.

- 5) The pressure, P , is 748.37 mm Hg and the relative humidity, h , is 46.92 %.

$$\rho_a = \left[\frac{0.46460 [P - (0.0037960 h e_s)]}{273.15 + t} \right] / 1000$$

Calculate the air density in g/cm^3 using the above equation and your result from question 4.

- 6) The basis for advanced weighing designs is the double substitution method, which usually incorporates air-buoyancy corrections. The general procedure involves weighing a known standard, S, then an unknown, X, the unknown plus a sensitivity weight, X + sw, and, finally, the standard plus the sensitivity weight, S + sw. The formula for calculating the true mass of X, M_x , is provided below, along with the data from a single set of measurements and densities of the various materials. The true mass of the standard and sensitivity weight, M_s and M_{sw} , respectively, are also included in the below table.

O_1	0.00	ρ_a	0.00118432 g/cm ³	M_s	10.00439 g
O_2	-0.01	ρ_x	8.03 g/cm ³	M_{sw}	30.0625 mg
O_3	30.06	ρ_{sw}	7.95 g/cm ³		
O_4	30.04	ρ_s	8.00 g/cm ³		

$$M_x = \frac{M_s(1 - \rho_a/\rho_s) + \left[\frac{(O_1 - O_2) + (O_4 - O_3)}{2} \right] \left[\frac{M_{sw}}{O_3 - O_2} (1 - \rho_a/\rho_{sw}) \right]}{(1 - \rho_a/\rho_x)}$$

Circle all of the terms in the equation above that are approximately one.

- 7) Calculate M_x in grams using the above equation.

- 8) The uncertainty, u , in a measurement is the square root of the sum of the square of the uncertainties for each component in the calibration. The expanded uncertainty, U , is the above uncertainty multiplied by a coverage factor, k . In this particular case the uncertainty contains three components: $u_p = 0.00323$ mg, $u_s = 0.00068$ g, $u_{abc} = 0.000428$ mg, and $k = 2.17$

$$U = ku$$

$$u = \sqrt{u_p^2 + u_s^2 + u_{abc}^2}$$

What are the units underneath the square-root sign?

- 9) Calculate the expanded uncertainty in mg.