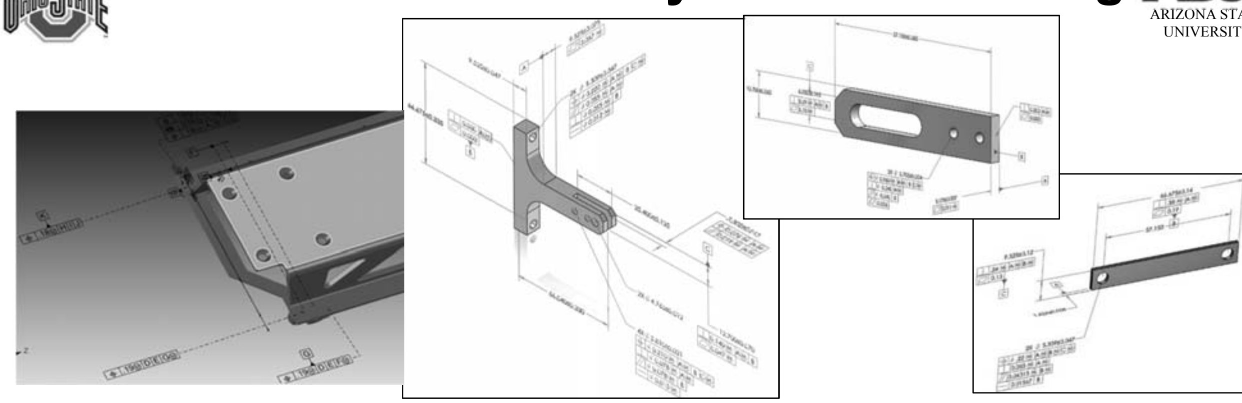
Beyond Symbology: Mathematical Models for GD&T

Joseph Davidson[1] & Jami J. Shah[2]
[1]Arizona State University, Tempe, AZ 85287
[2]The Ohio State University, Columbus, OH 43210

shah.493@osu.edu
j.davidson@asu.edu

Funding provided by NSF Grants
DMI-9821008,
DMI-0245422
CMMI-0700878
CMMI-10036128
DARPA AVM HR0011-12-C-0075
DMDII 14-05-02
DMDII 16-01-02

Geometric Variability in Manufacturing



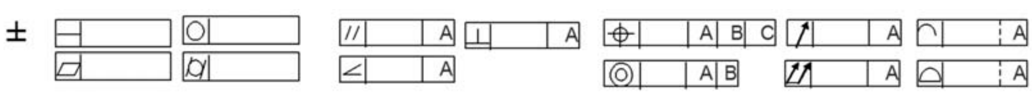
- **Geometric dimensions & tolerances are of concern in all aspects of product development.**
- **Miscommunication and misinterpretation can result in low acceptance rates or expensive rework.**
- **Math models can not only prevent such problems but also provide a better medium for PMI information transfer between modern digital tools: CAD, CAM, CAE, CMM…**

GD&T standards vs Math Models

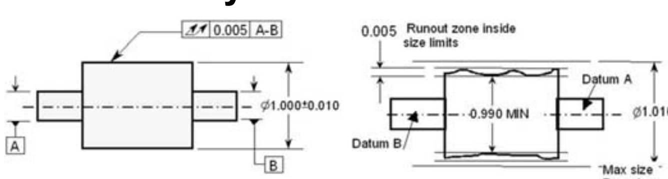
- **In engineering practice, tolerances are specified using national and international GD&T standards, such as ASME Y14.5M and ISO 1101.**
- **These standards are not based on any math foundation; they are a set of symbols and conventions based on technical drawings**
- **STEP PMI data models are based on these very same conventions**
- ***This is not conducive for model based engineering!***
- **Over the past decades, many models have been proposed for tolerance representation & analysis, broadly classified as:**
  - PARAMETRIC MODELS [Hillyard et al; Gossard et al]
  - OFFSET ZONES [Requicha et al]
  - VARIATIONAL SURFACES [Martinsen 93, Turner 90]
  - VECTOR SPACES [Turner & Wozny 90, Clement et al, Davidson & Shah et al]
- **If a model is to gain acceptance, it must be consistent with engineering practices that were driven by GD&T standards**

Note that Y14.5.1 does not propose any particular math model, but attempts to give rational explanations

ASME Y14.5 Conventions: A Quick Look



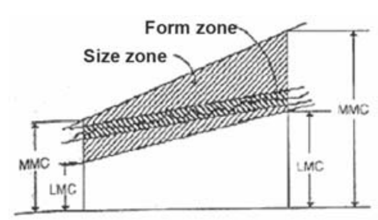
Geometric variations have been decomposed into specific types because they affect function & assembly in different ways

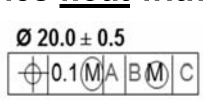


Each variation is represented by a zone whose shape depends on the toleranced feature type;

Zone size depends on tolerance value and modifiers;
Zone location depends on tolerance type and datums

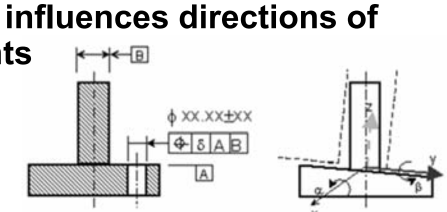
Certain tolerances are refinements of others; some zones <u>float</u> within other zones (Rule#1)



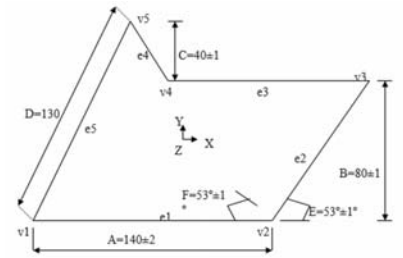


<u>Bonus Tolerance & Shift</u> :
Material conditions (MMC, LMC) can enlarge position tolerance zones by the difference between MMC (or LMC) and actual size

<u>Datum order</u> influences directions of measurements



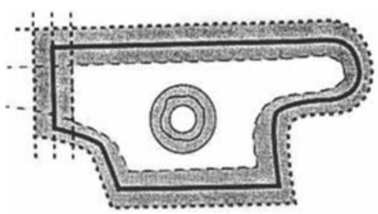
Survey of GD&T Math Models



PARAMETRIC MODELS

Closely related to parametric CAD: uses the same set of parameters and constraints as those used in geometry construction; Tolerances are +/- variations applied to dimensions; the math model is the constraint set

Cannot support form tolerances, datum ref frames, directional relations,…

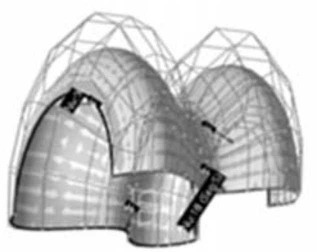


OFFSET ZONE MODELS

Tolerance zone created by Boolean subtraction of volumes obtained by offsetting a part's boundaries by equal amounts on either side

Cannot distinguish between variation types (size, form, orientation, position)

No DRFs, material or other modifiers



VARIATIONAL SURFACES

NURBS/B-spline Control Points are assigned tolerance values

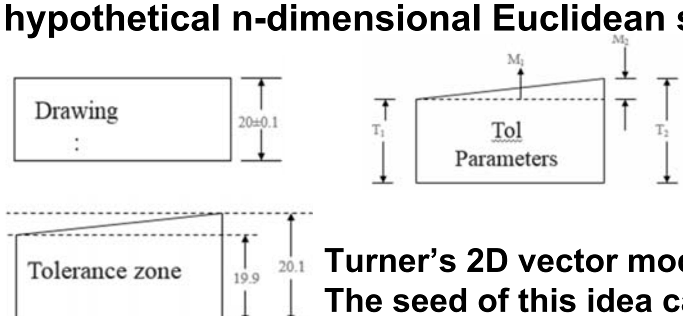
Non-intuitive- no explicit relation between CPs and GD&T

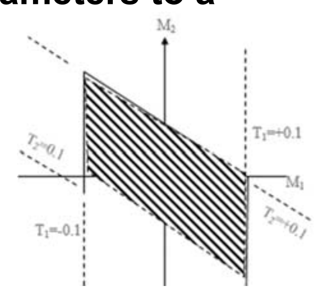
Cannot differentiate between different tolerance classes

No DRFs, modifiers, zones

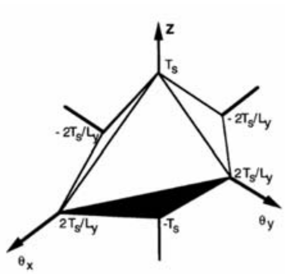
Vector Space Models (Early work)

The basic idea is to map geometric variation parameters to a hypothetical n-dimensional Euclidean space





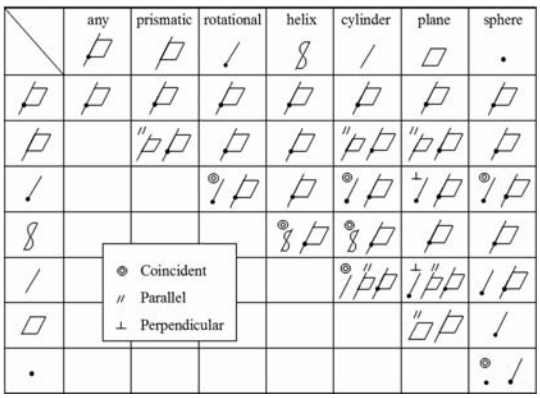
Turner's 2D vector model of size:
The seed of this idea can be found in Josh Turner' work from the 90's; but it was not pursued further

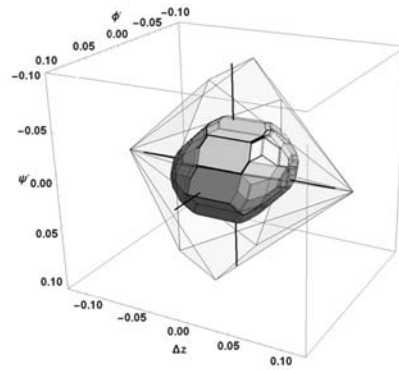


Whitney's 3D vectorial tolerance model for size
Max boundary in a "kinematic parameter space"

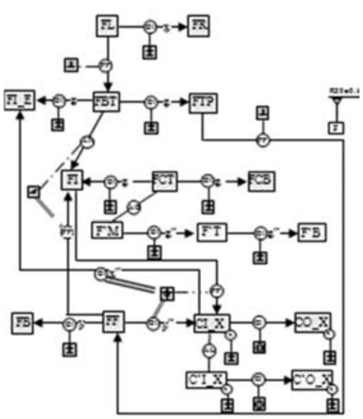
Vector Space Models SoA

- Two groups have advanced these types of models to a high degree of maturity:
  - T-maps: Developed at Arizona State since 1995
  - Deviation Space & TTRS at U. Cachan, Annecy (France)
- Treats geometric entities (points, line, planes) as if they were rigid bodies with DOFS
- Models how each DOF of each entity is controlled
- Generates convex regions in n-dimensions to represent ALL combinations of tolerance variations

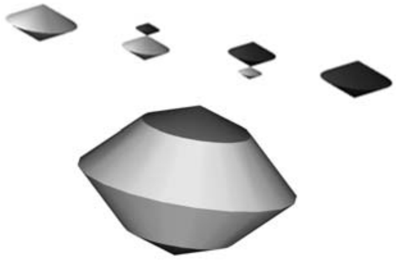




ASU GD&T Math Model*



TOPOLOGICAL MODEL (DoF algebra; datum flow):
- Similar to topology in solid modeling
- Models relationships between all feature control frames, datum reference frames (DRF) and their precedence (datum flow chain)
- provides basis for geometric validation of D&T scheme, loop detection for analysis and DoFs
- Supported by DoF algebra

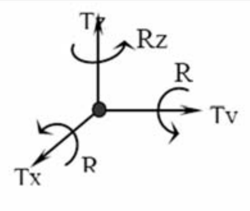
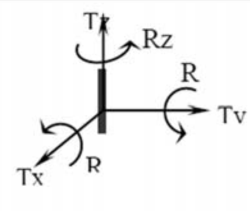
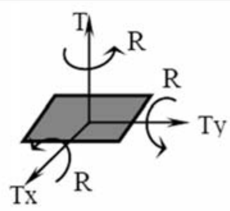


METRIC MODEL (T-Maps):
- models the composite quantitative effect of all tolerances on a given feature
- interaction of size, form, orientation, position is clearly identified
- Rule #1 is embedded in the formulation
- relative volumes of regions can be used to study trade-offs in tolerance allocation (size vs form vs orientation..)

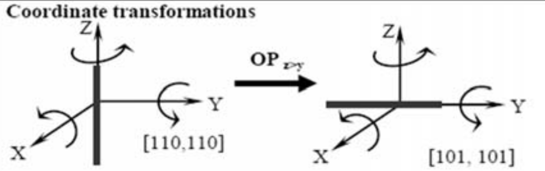
* US Patent No. 6,963,824

Topological Model: DoF Algebra

- DRFs and TRFs are clusters of points, lines and planes with different geometric relations to each other (coincident, //, ⊥, …)
- DoF Algebra includes symbolic ops to determine free and invariant DoFs of entity clusters.
- This algebra was validated by applying it to all cases in the Y14.5.1.

	Point	Line	Plane
			
X_{fdof}	$(T_x\ T_y\ T_z, - - -) \rightarrow (111,000)$	$(T_x\ T_y -, R_x\ R_y\ -) \rightarrow (110,110)$	$(- -\ T_z, R_x\ R_y\ -) \rightarrow (001,110)$
X_{inv}	$(- - -, R_x\ R_y\ R_z) \rightarrow (000,111)$	$(- -\ T_z \ldots R_z) \rightarrow (001,001)$	$(T_x\ T_y -, -\ Rz) \rightarrow (110,001)$

Coordinate transformations



We define the CS Transform operator **OP** $_{i>j}$ as swapping of the i and j element values for both TDof and RDoF.
Thus, **OP** $_{z>y}$ { [110,110]} = [101,101].
This is valid for any entity or cluster and for active or invariant DoF vector.

Algebraic Operators

Union Operation ∪			Intersection Op ∩		
a_i	b_i	ai ∪ bi	a_i	b_i	ai ∩ bi
1	1	1	1	1	1
1	0	1	1	0	0
0	1	1	0	1	0
0	0	0	0	0	0

Combining DoFs for clusters

$[X_{fdof}] = [A_{fdof}] \cup [B_{fdof}]$;
$[X_{inv}] = [A_{inv}] \cap [B_{inv}]$

Algebraic Relations

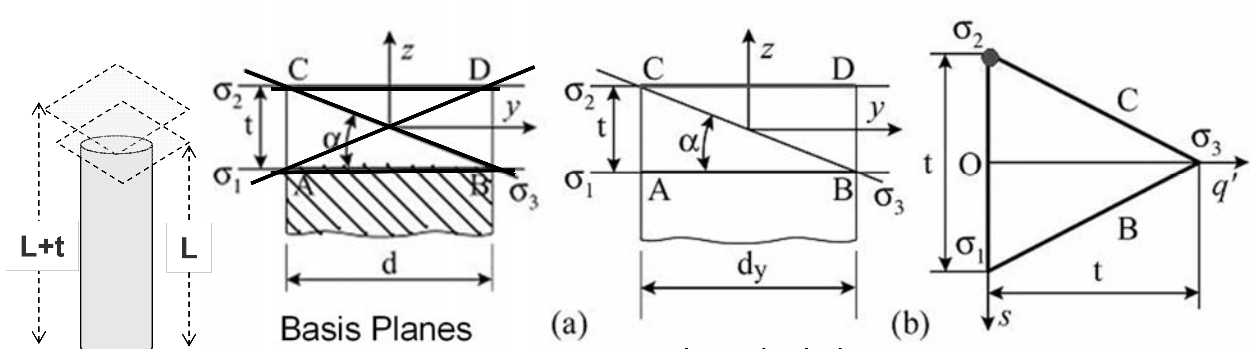
- [A] ∪ [B] = [B] ∪ [A] ………………Commutative relation
- $[A_{fdof}] \cap [A_{inv}]$ = [∅]=[000,000] …….Null set
- $[A_{fdof}] \cup [A_{inv}]$ = [**I**]=[111,111]…… "Identity" vector
- $[A_{inv}]$ = RCP $\{[A_{fdof}]\}$ …. ……… Reciprocal relation (or Ā)
- +Standard Associative, Distributive and Idempotence relations

- **DoF algebra models datum flow chains, DRF combinations and tolerance classes**
- **The controlled DOFs are the intersection of the DOFs of three tolerance elements.**
- **No matter what the target cluster is, the DOF vector of target entity is one of six combinations.**

No.	Target	DRFs	Tol. Class	Constrained DOFs
1	(111,000)	(111,000) (111,110) (111,111)	(111,111)	(111,000)
[illegible]	[illegible]	(110,110) (111,111)	[illegible]	[illegible]
6	(111,111)	(110,110) (001,110)	(000,111)	(000,110)
		(111,111)	(111,111)	(111,111)

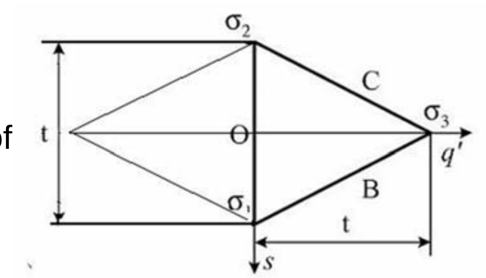
Tolerance Maps for size:
planar feature

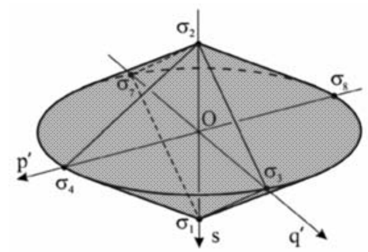
Cylindrical bar cross-sections



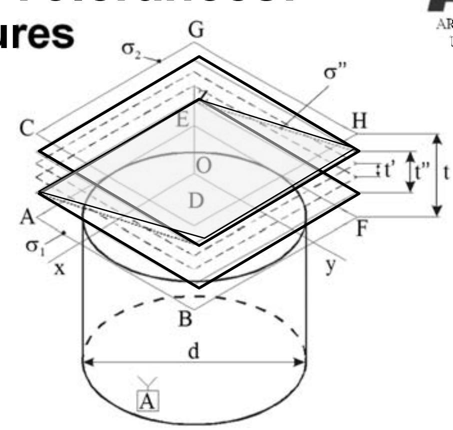
Any plane (point) $\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3$

Cross section of planar T-map©





Form & Orientation Tolerances:
Planar Features

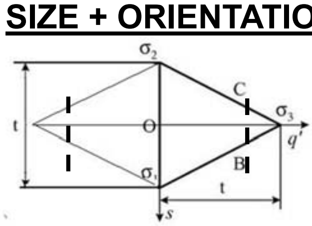


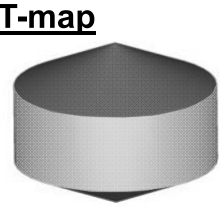
FLOATING ZONES

ORIENTATION zone (*t*") translates ↑↓ can rotate about *x*- or *y-axes*

FORM zone (*t'*) translates ↑↓ and rotates about *x*- or *y-axes*

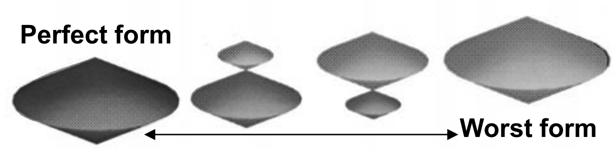
SIZE + ORIENTATION T-map





Addition of orientation tol *t*" to size reduces the allowable tilt
Orientation T-map can be obtained from size by truncating the σ_3 axis

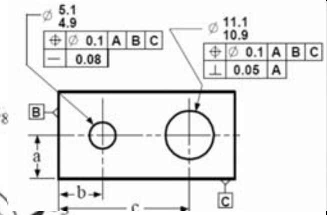
SIZE + FORM T-map



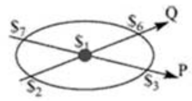
As per Y14.5 Rule#1
- **Worst form occupies the entire zone**
- **Perfect form occupies none**

Therefore, size + form is modeled by splitting into two planar T-maps that together must conform to size map

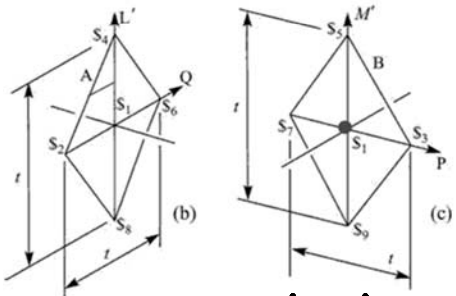
Tolerance Maps For Lines: 4-D Solid of Points



2D cross-sections of the T-Map

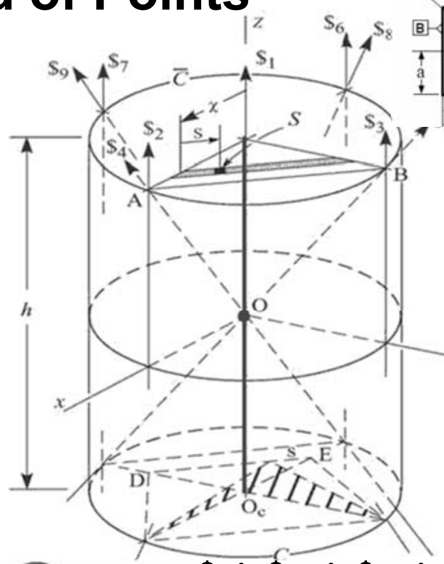


$\lambda_4 = \lambda_5 = 0$

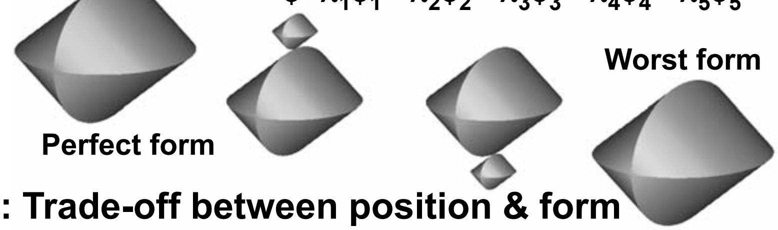


$\lambda_3 = \lambda_5 = 0$

$\lambda_2 = \lambda_4 = 0$



$\$ = \lambda_1\$_1 + \lambda_2\$_2 + \lambda_3\$_3 + \lambda_4\$_4 + \lambda_5\$_5$

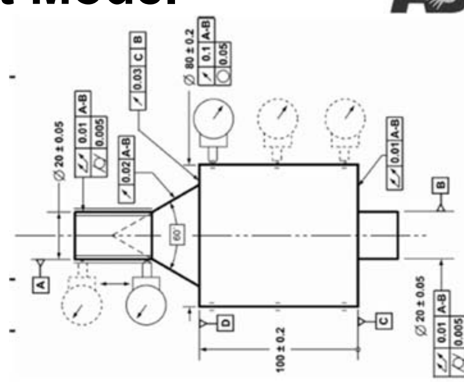


Perfect form

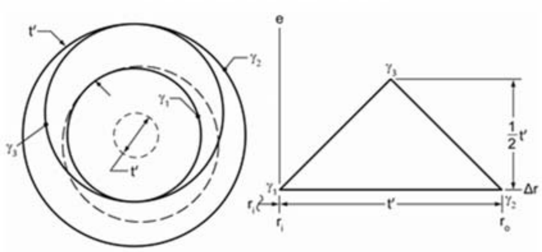
Worst form

3D cross-sections: Trade-off between position & form

Circular Runout Model

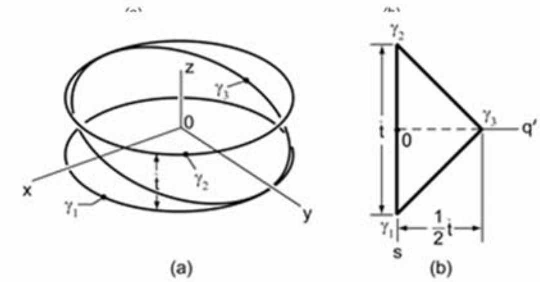


- Circular runout is a composite tolerance that controls both circularity and concentricity (position), independent of size
- Applied to any axisymmetric X-sec



- circular X-sec, involves two variables: circularity (annular zone) + eccentricity

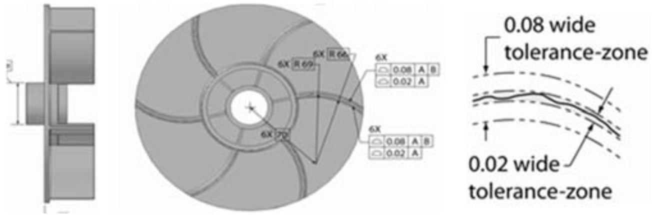
(a) An annular tolerance-zone of amount t′ which lies between the inner and outer boundaries γ_1 and γ_2 of radii r_i and r_o, respectively. (b) Its 2D T-Map



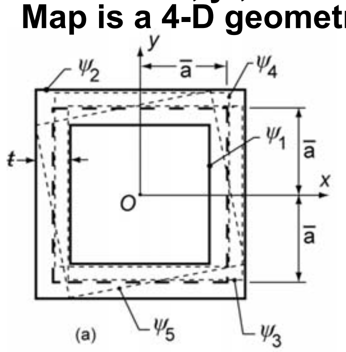
- planar (end) involves two variables: linear offset + angle

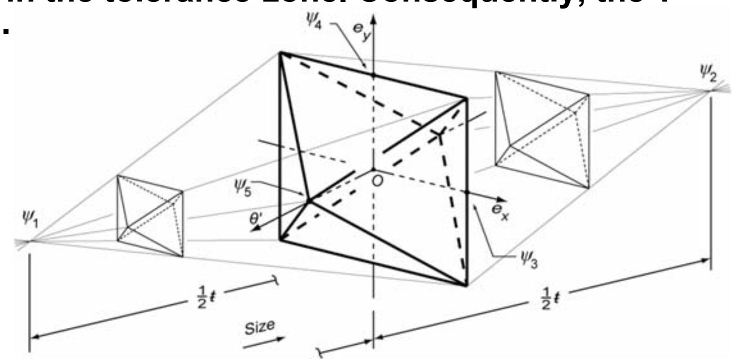
(a) A cylindrical tolerance-zone of height t̀ which lies between the upper and lower boundaries of y1 and y2. (b) Its 2D T-Map.

Line Profile: parametric model



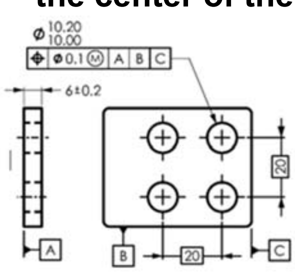
- Profile tolerances control the shape, size, and position of complex features, e.g. turbine blades and pump vanes.
- For line profiles, four variables are required to identify a variation of the theoretical shape within its tolerance-zone.
  - <u>Example: A square line-profile</u>
  - For line profiles, each point in the T-Map represents one square with a given size and x-, y-, and θ-position in the tolerance-zone. Consequently, the T-Map is a 4-D geometric shape.

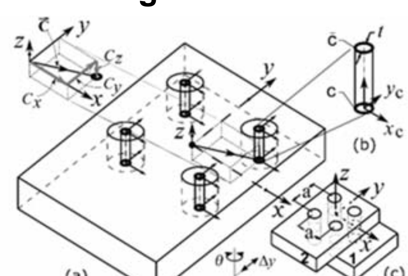


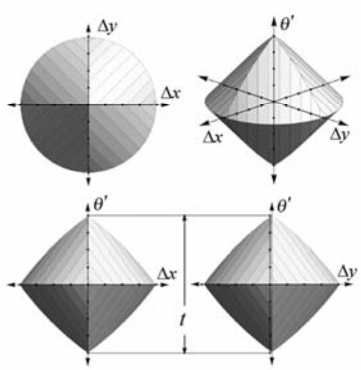


T-maps for Feature Patterns

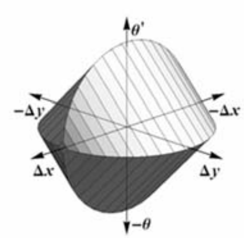
- For each part, the T-Map that models limits to relative displacements between hole patterns formed by transforming the T-Map for each individual feature in its local frame of reference to a global frame that is at the center of the pattern.







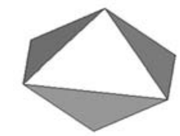
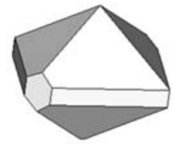
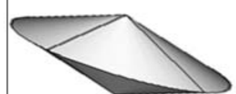
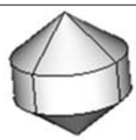
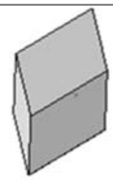
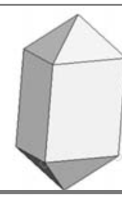
Part-level T-Map for a circular (square) pattern of four holes



Part-level T-Map for opposite holes, represented in frame *xyz*.

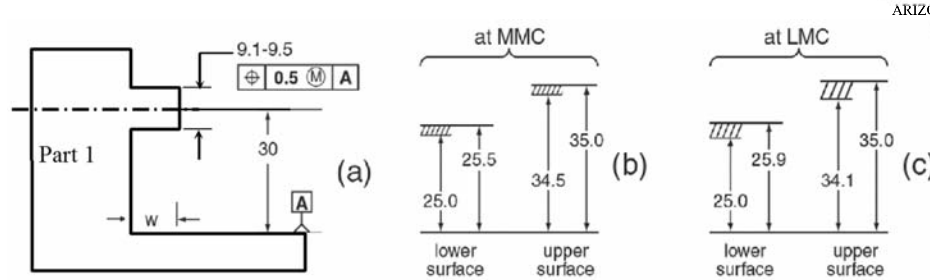
T-map Catalog: Sample page

More than 50 T-map models have been developed so far based on combinations of target feature, tolerance type and datum type

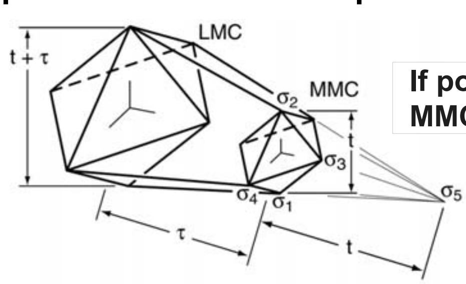
T-map	Geometry, tolerance, datum	T-map	Geometry, tolerance, datum
	Geom: Rect bar; plane Tol class: size Datum: none		Geom: Rect bar; plane Tol class: size + orient Datum: planar face
	Geom: Round bar; plane Tol class: size Datum: none		Geom: Round bar; plane Tol class: size + orient Datum: offset axis
	Geom: Round bar; plane Tol class: size + orient Datum: planar face		Geom: Planar circular face Tol class: circular runout Datum: axis
	Geom: traing bar; plane Tol class: size Datum: none		Geom: Rect bar; plane Tol class: size + orient Datum: two datums

Material Modifiers in T-map models

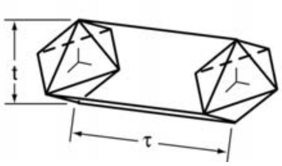
t = pos. tol
τ = size tol.



4-D T-Maps: size is the 4th dimension
The dipyramid now is the T-map for position of the medial plane.



If pos tol uses MMC modifier



If pos tol uses RFS modifier

Hyper-Volume computation
Hyperpyramid of dimension n

$$_nV = \frac{1}{n}\left({}_{n-1}C\right)h$$

$$_4V_f = \frac{1}{4}\left\{{}_3C_{t+\tau}(t+\tau) - {}_3C_t\, t\right\} = \frac{1}{6}\left\{(t+\tau)^4 - t^4\right\}.$$

Hyperprism of dimension n

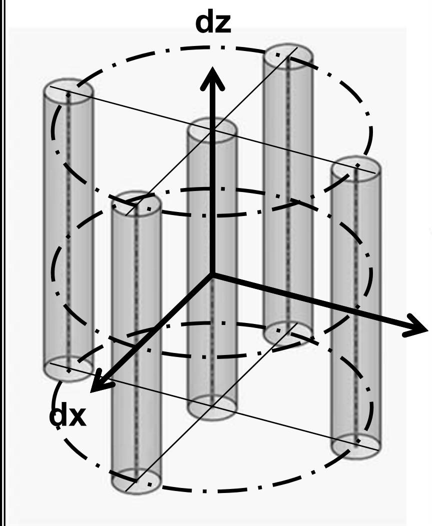
$$_nV = {}_{n-1}C h$$

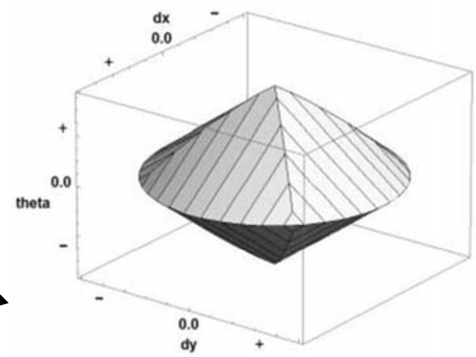
$$_4V_p = {}_3C_t\tau = \frac{2}{3}t^3\tau$$

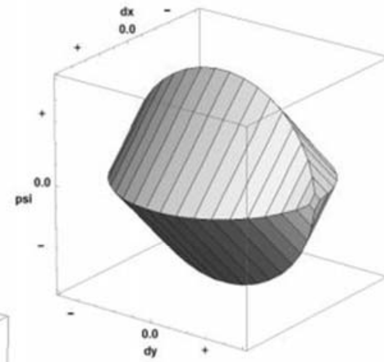
Insight: if t = τ $V_{MMC} = 3.75\ V_{RFS}$

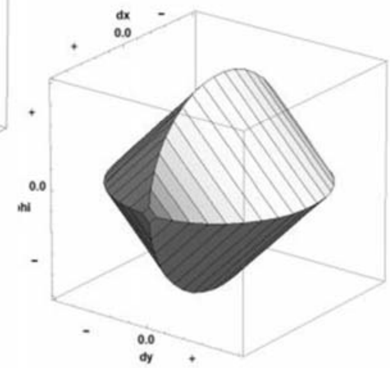
T-maps for Feature Patterns

- Pattern T-map is similar to Axis T-map but with one added dimension viz. theta.
- Theta represents possible rotation about dz axis.

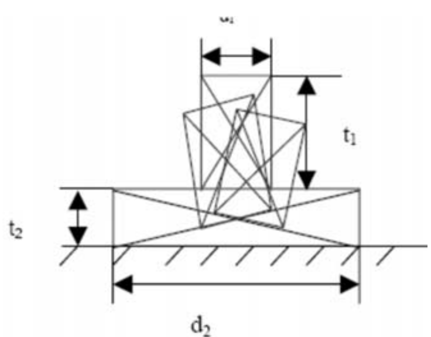




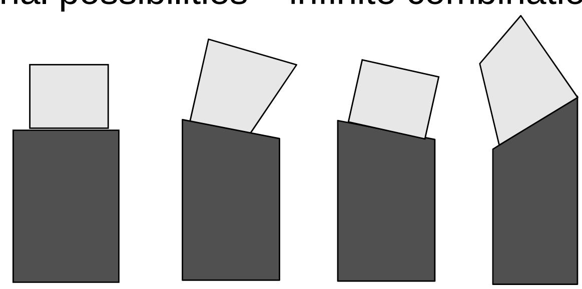


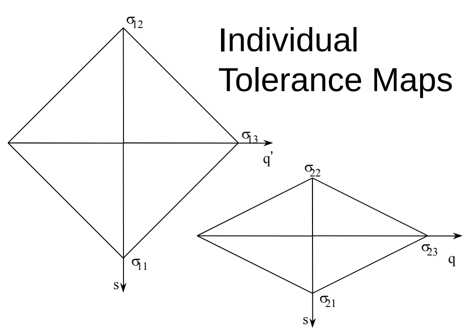


Tolerance Analysis with T-maps: Minkowski Sums



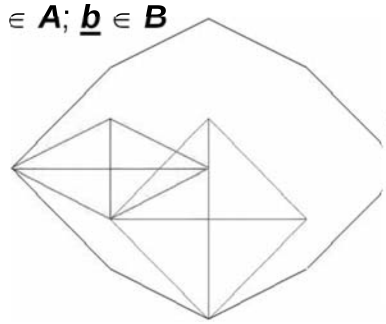
Variational possibilities – infinite combinations



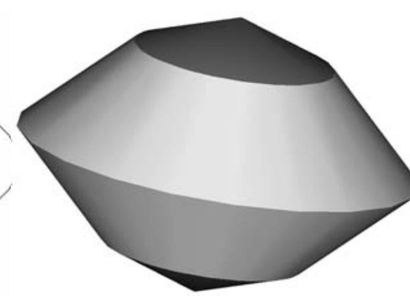


Individual Tolerance Maps

Minkowski sum: C = U*c*, where ***c*** = ***a*** + ***b*** and ***a*** ∈ ***A***; ***b*** ∈ ***B***

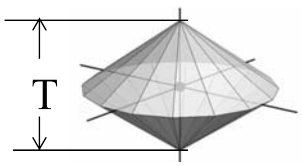


Accumulation map

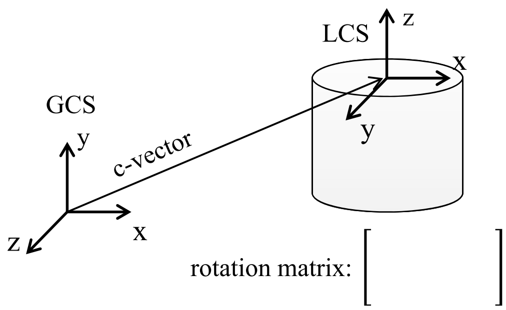


Transforming T-maps from local to global CS

- **Each feature's variation in the target gap is evaluated one by one by generating Local T-Map for that feature and then transforming it to Global coordinate frame**



T-Map in LCS



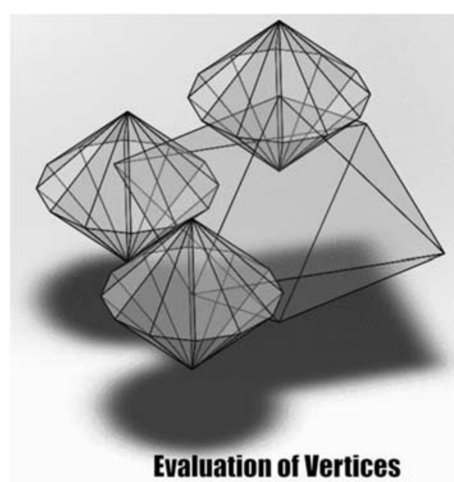
T-Map transformed from LCS to GCS (shear)

T = Tolerance specified on feature

Analysis with T-maps: Minkowski sum

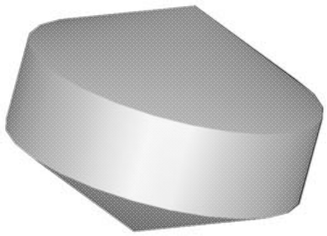
- **Minkowski Sum of two T-Maps A and B is a vector sum of its vertices in 6 dimensional space. One vertex is a six dimensional vector. Both T-Maps are represented with set of vectors.**
- Every vector in T-Map A is added to all vectors in T-Map B using vector sum operation.
- Minkowski Sum thus produced is represented as A ⊕ B.
- Internal points so produced are not useful for further analysis and are omitted by forming a convex hull out of generated vectors.
- Further redundant vertices are eliminated using qhull to produce a convex hull. Which represents Minkowski Sum of two Tmaps.

	T-map 1	T-map 2	Vector Addition	Reduced#
# Vertices	6	14	14 x 6 = 84	22

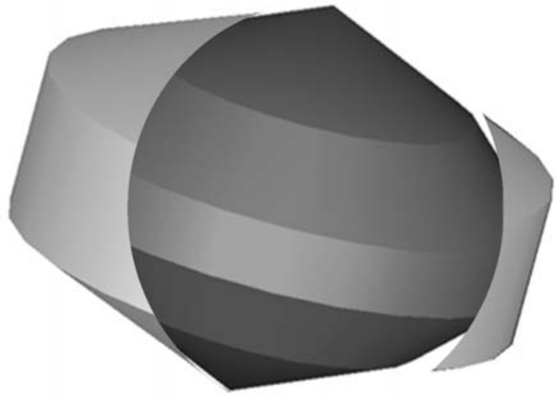


Worst case analysis with T-maps:
Functional & Accumulation Maps

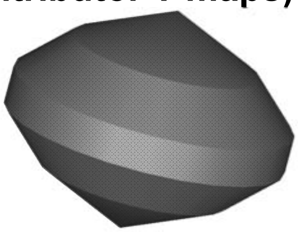
Functional Map
(e.g desired clearance)



Does accumulation map fit inside functional map?

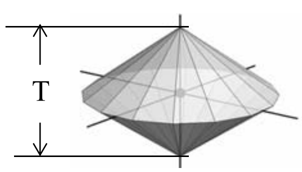


Accumulation Map (variations of all contributors : Minkowski sum of contributor T-maps)



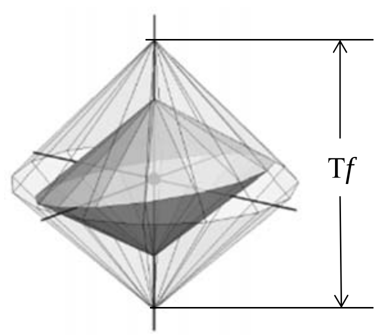
Statistical Analysis with T-maps

- **Sensitivity of each feature to variation in target gap is evaluated one by one by generating Local T-Map for feature, transforming it to Global coordinate frame where target gap is located and then by fitting functional T-Map over Global T-Map.**
- **Sensitivity is ratio of tolerance associated with functional T-Map to the tolerance on feature**



T-Map in LCS

T-Map transformed from LCS to GCS



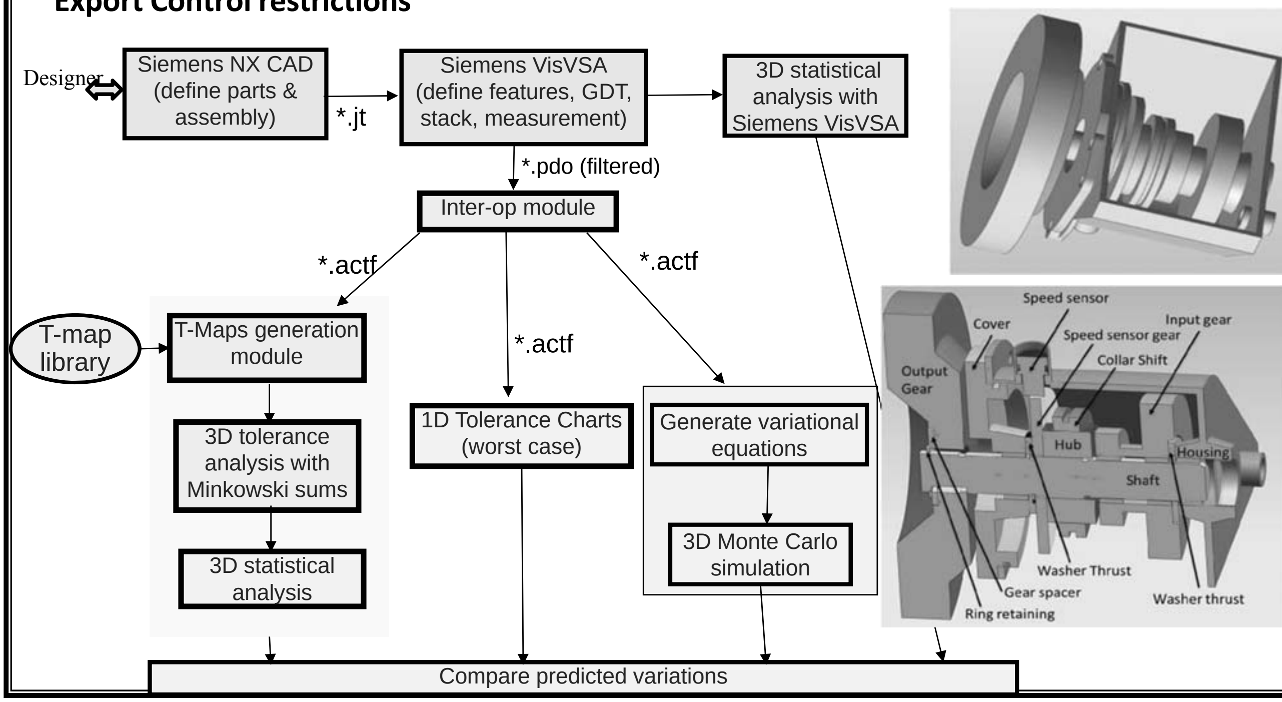
Functional T-Map fitted over transformed T-Map

T = Tolerance specified on feature
T_f = Functional tolerance at target gap contributed by feature
S = Sensitivity of feature to target gap
= T_f / T

Benchmarking T-maps

ARIZONA STATE UNIVERSITY

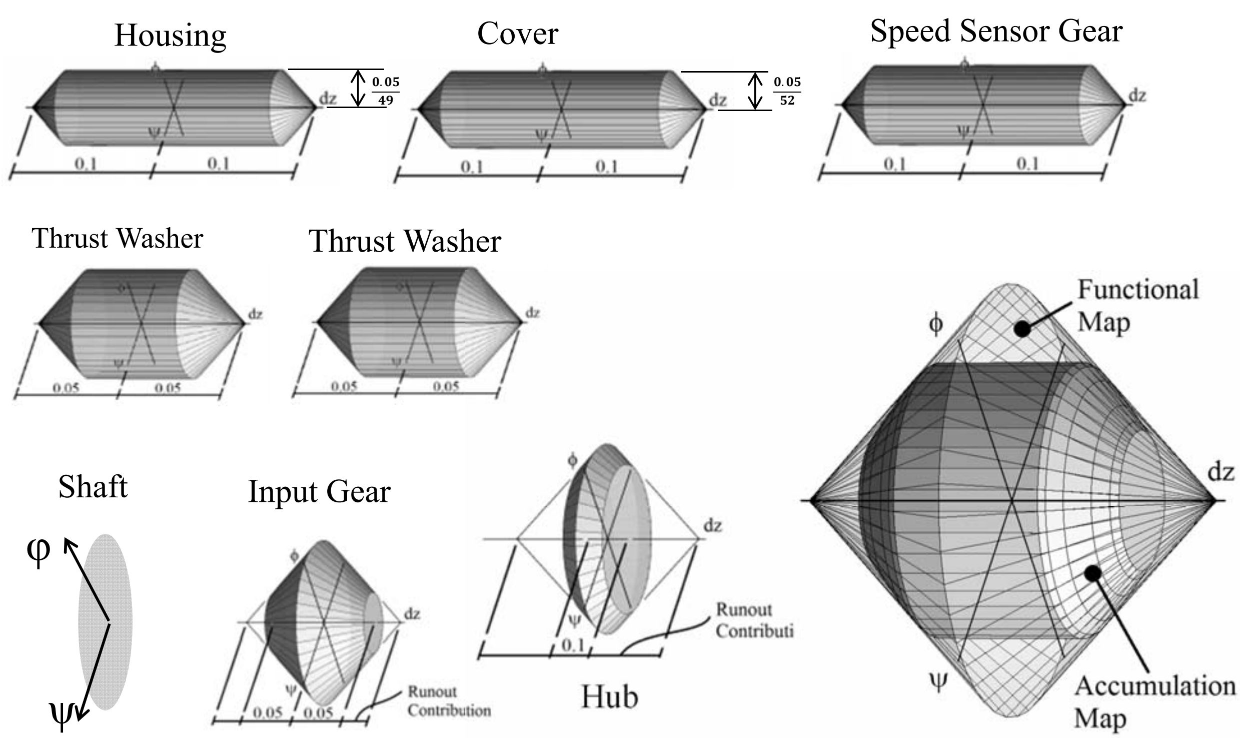
- T-map generation, transformation, accumulation procedures were applied to complex assemblies under a DMDII project
- This surrogate assembly will be used to illustrate the process because of RollsRoyce & Export Control restrictions



Worst Case analysis for Assembly

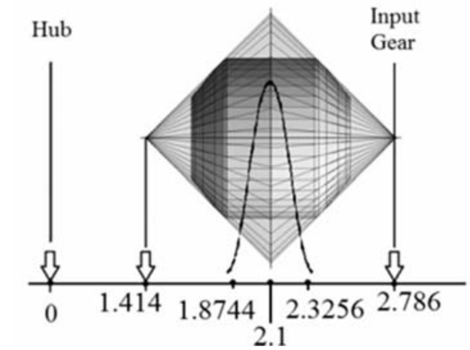
T-Maps for features in stack for Gap 1

ARIZONA STATE UNIVERSITY



Sensitivity Analysis for Assembly

Sr.No.	Part	Tol. Type	Tol. Value T	Sensitivity S	Mean μ	SD (Six Sigma) σ
1	Hub	Size	0.10	1.03	0	0.0172
2	Shaft	Runout	0.10	1.52	0	0.0253
3	Speed Gear Sensor	Size	0.20	1.00	0	0.0333
4		Parallelism	0.05	0.00	0	0.0000
5	Washer Thrust	Size	0.10	1.00	0	0.0167
6		Parallelism	0.05	0.00	0	0.0000
7	Cover	Size	0.20	1.00	0	0.0333
8		Parallelism	0.05	0.00	0	0.0000
9		Runout	0.10	0.13	0	0.0022
10	Housing	Size	0.20	1.00	0	0.0333
11		Parallelism	0.05	0.00	0	0.0000
12		Runout	0.09	0.13	0	0.0020
13	Input Gear	Size	0.10	1.19	0	0.0198
14		Runout	0.03	0.93	0	0.0046
15	Washer Thrust	Size	0.10	1.00	0	0.0167
16		parallelism	0.05	0.00	0	0.0000




```
### distribution parameters ###
mean       -2.07456
variance          0.00193292
absolute deviation       0.0348872
skewness          0.0749801
kurtosis          0.182128
minimum value     -2.24407
maximum value     -1.8882
```

$$V = \sum S * T$$

Where V = variation in clearance (Random variable)
S = Sensitivity (Constant)
T = Tolerance (Random variable)

$$V = 1.03T1 + 1.52\ T2 + T3 + T7 + 0.13\ T9 + T10 + 0.13\ T12 + 1.19\ T13 + 0.93\ T14 + T15$$

Benefits of the T-maps Math Model

- **Models distinctly all types of tolerances incuding freeform profile**
- **Models tolerance variations in 3D**
- **Represents floating zones for orientation and form**
- **Accounts for material conditions: Models the tradeoff between size/orientation and position (bonus)**
- **Accounts for the choice and order of datum reference frames (DRFs)**
- **Can be used for worst case or statistical analysis**
- **Metrics can be used to aid in the allocation of tolerances in a stackup**
- **Is extendible to other features and to clusters of features**

- **All tolerance-Maps are convex bodies: allows use of some standard computational algorithms**
- **Is in Euclidean space: metric computations can be made using standard formulae**