Obstacle detection using limited measurements of scattered waves

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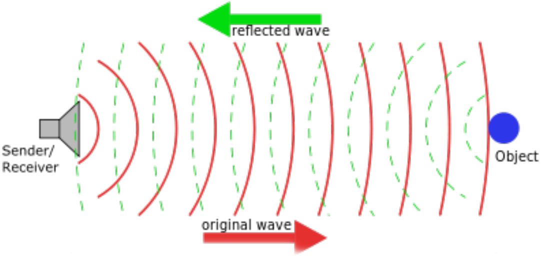
NIST ACMD Seminar

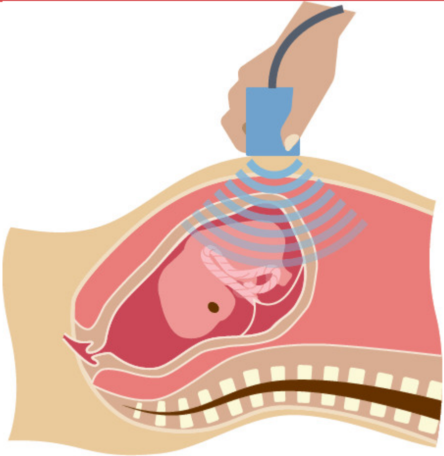
What is inverse scattering theory?



Determine information about unknown obstacles based on how acoustic or electromagnetic waves scatter off of them.

By "scattered" I mean reflected, transmitted, and absorbed.

Ultrasound

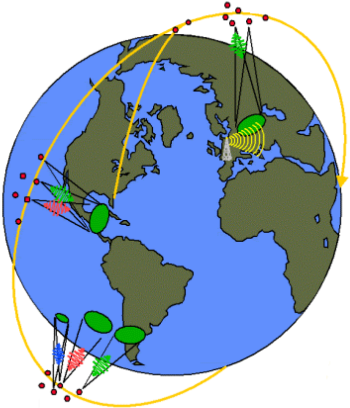


Acoustic waves can characterize human tissue.

Testing airplane canopies

Quickly finding flaws can save millions of dollars.

TechSat 21 Project



RF Imaging with distributed satellites.

Challenges

- Vastly different material properties (human body vs. state-of-the-art materials)
- Orders-of-magnitude different sizes (landmasses vs. microscopic cracks)
- Cost to probe (satellites vs. ultrasound machines)

Can we design general techniques which are useful at all of these scales?

How expensive are these techniques?

Are we confident in our results?

Different approaches to inverse problems

Iterative Methods	Statistical Methods	Direct Methods

Iterative Methods

- "Guess" a solution and check against collected data
- Solve large non-linear optimization problem
- Need to understand physical system

Statistical Methods

- Use Bayes rule to incorporate prior information into solution
- Typically uses an iterative method within the algorithm

Direct Methods

- Use mathematical properties of system non-iteratively
- Fast, but need lots of data

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Direct Methods

- Use mathematical properties of system non-iteratively
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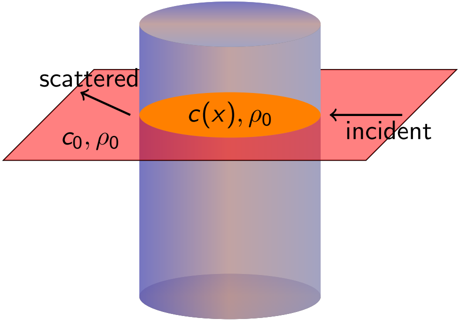
Today's topics

- Introduction to wave scattering theory
- Ways to use understanding of system to reduce measurement requirements
- Uncertainty in reconstructions
- Bayesian inversion

Chapter One
Scattering Theory

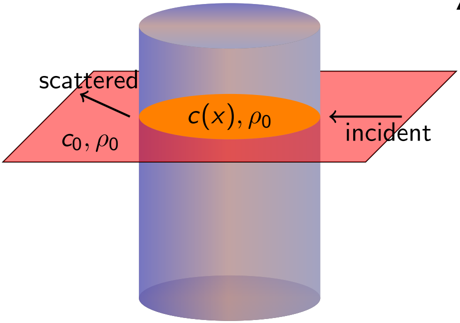
Simple physical setting



- Scattering from an infinite cylinder
- Acoustic speed of sound, c, and pressure, ρ, are constant outside object
- Speed of sound changes inside object

Acoustic wave scattering of a time-harmonic incident field from an penetrable object.

Simple physical setting



$$\Delta u_y^s(x) + k^2 n(x) u_y^s(x) = -k^2 n(x) u_y^i(x)$$

$$\Delta u_y^i(x) + k^2 u_y^i(x) = 0$$

$$\lim_{|x|\to\infty} |x|^{1/2} \left(\frac{\partial u^s}{\partial \nu} - iku^s \right) = 0,$$

k is the wave number

$n(x)$ is related to physical constants

u^s is the scattered acoustic field

u^i is the incident plane wave

Acoustic wave scattering of a time-harmonic incident field from an penetrable object.

Simple physical setting

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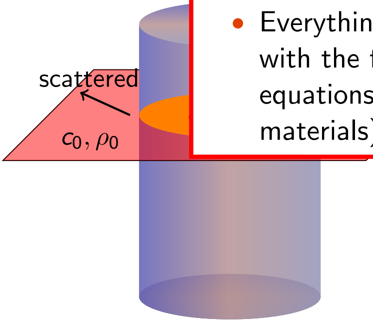


- Everything that follows can be done with the full system of Maxwell's equations (for realistic electromagnetic materials).

$$\ldots s \Big) = 0,$$

k is the wave number

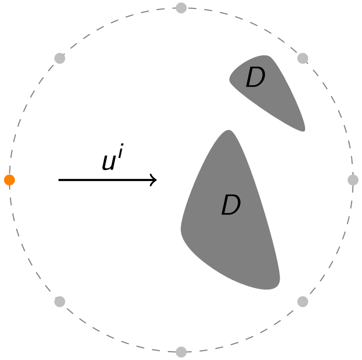
$n(x)$ is related to physical constants

u^s is the scattered acoustic field

u^i is the incident plane wave

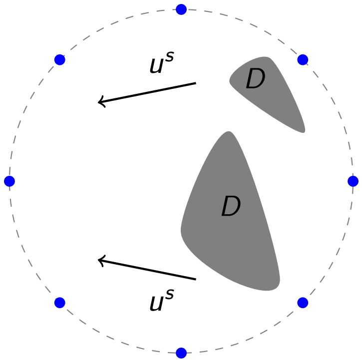
Acoustic wave scattering of a time-harmonic incident field from an penetrable object.

Multistatic Data collection



Determine material parameters by measuring how the material affects an incident wave.

Multistatic Data collection



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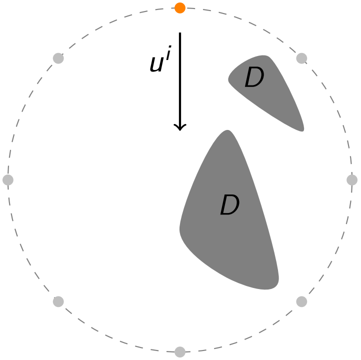

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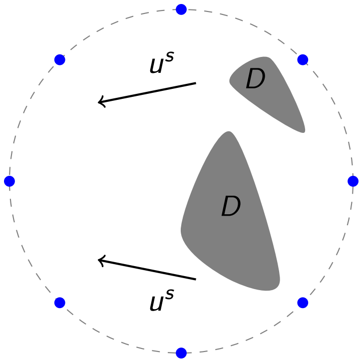

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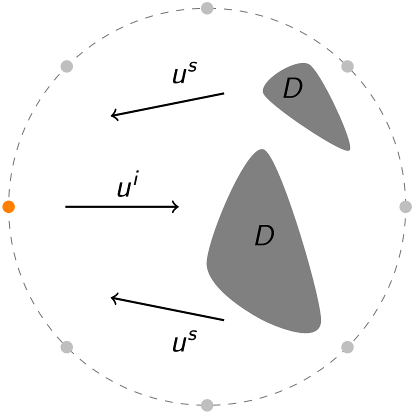
Multistatic Data collection



- Useful in medical settings or testing material properties of small objects
- Lots of reconstruction algorithms can be used with this type of data
- Impractical for most applications

Can we do better?

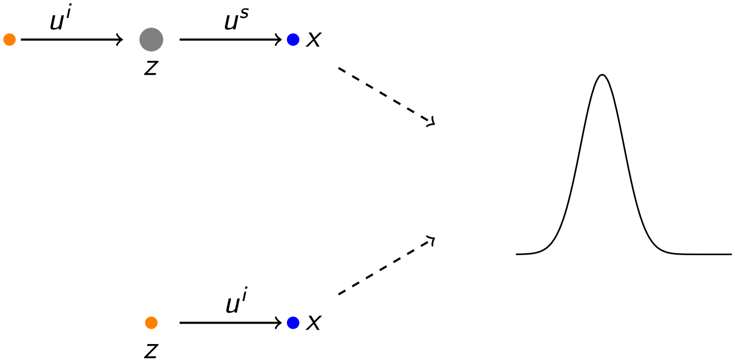
Direct inversion algorithms

- Assume only the location of object is needed (e.g., looking for a crack or a tumor).
- Based on model, derive an indicator function $I(z)$, depending on coordinates, so that

$$I(z) = \begin{cases} 0 & z \notin \text{object} \\ 1 & z \in \text{object}. \end{cases}$$

- $I(z)$ must be easy and fast to compute from scattered field data.

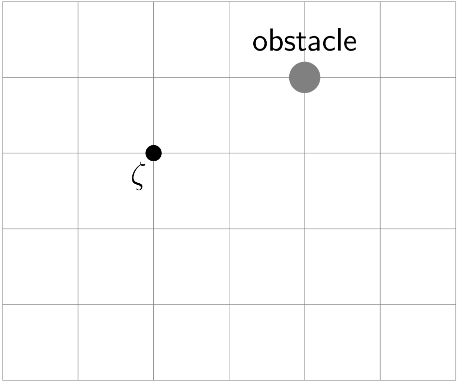
Intuition behind reconstruction algorithm



The scattered field from a small object at point z is proportional to an incident field emitted from the point z.

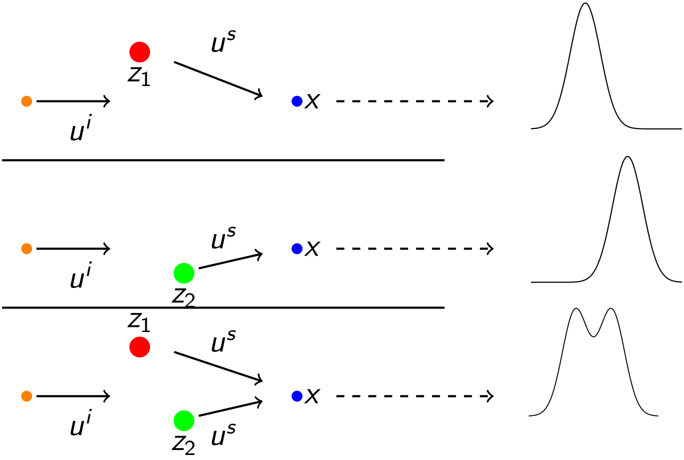
Reconstruction from one obstacle

- Collect u^s at receivers around obstacle.
- Generate a grid $\mathcal{Z}$ which contains the obstacle.
- For each point $\zeta \in \mathcal{Z}$ test if measured scattered field is proportional to an incident field coming from point ζ.
- This is possible if and only if ζ is the center of the obstacle.



$$u^s_y(x) g_\zeta = u^i_\zeta(x)$$

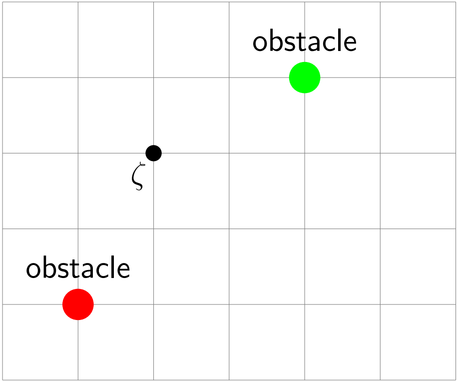
Intuition behind reconstruction algorithm



Scattered fields from multiple small objects behave *nearly* linearly.

Reconstruction of multiple obstacles

- Collect u^s at receivers around obstacle.
- Generate a grid $\mathcal{Z}$ which contains the obstacle.
- For each point $\zeta \in \mathcal{Z}$ try to match measured scattered field to an incident field coming from point ζ.
- The is possible if and only if ζ is the center of the obstacle.

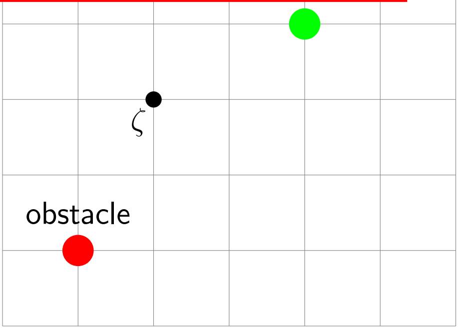


$$\int_{\Gamma_i} u_y^s(x) g(y)\,\mathrm{d}s(y) = u_\zeta^i(x)$$

Reconstruction of multiple obstacles

Exactly the same algorithm as for one obstacle.

- C… around obstacle.
- Generate a grid $\mathcal{Z}$ which contains the obstacle.
- For each point $\zeta \in \mathcal{Z}$ try to match measured scattered field to an incident field coming from point ζ.
- The is possible if and only if ζ is the center of the obstacle.



$$\int_{\Gamma_i} u^s_y(x) g(y)\,\mathrm{d}s(y) = u^i_\zeta(x)$$

Range test

At the discrete level, we want to find g so that

$$Ng = u^i_\zeta, \tag{1}$$

where N is the matrix which approximates $\int_{\Gamma_i} u^s_y(x) g(y)\, \mathrm{d}s(y)$.
Let $N = U \Sigma V^H$ be a singular value decomposition of N.

Equation (1) holds if and only if

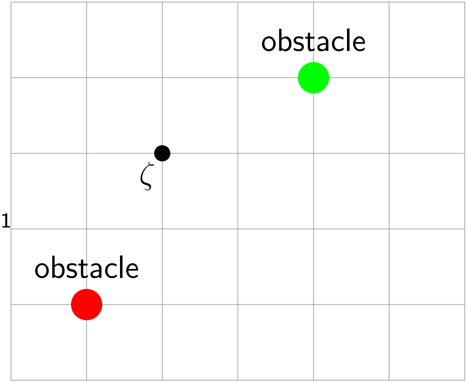
$$I(z) = \sum_{r=\text{number obstacles}+1}^{\text{number incident fields}} [U]_r u^i_\zeta \approx 0.$$

Reconstruction algorithm

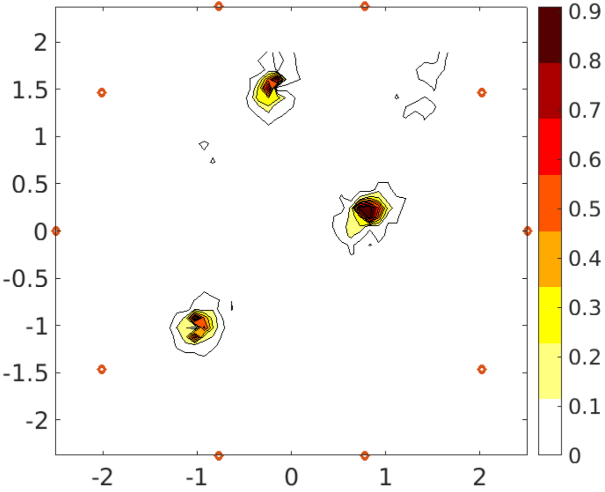
- Collect u^s at receivers around obstacle
- Generate a grid $\mathcal{Z}$ which contains the obstacle
- For each point $\zeta \in \mathcal{Z}$, plot

$$\left(\sum_{\text{number obstacles}+1}^{\text{number incident fields}} [U]_r u^i_\zeta \right)^{-1}$$

- Large values on grid indicate object



Simulated Reconstruction

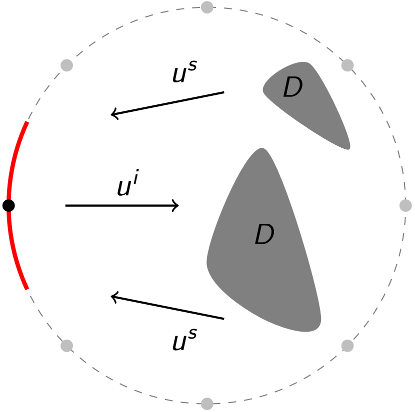


Measurement geometry

Reconstruction methods are independent of receiver geometry.

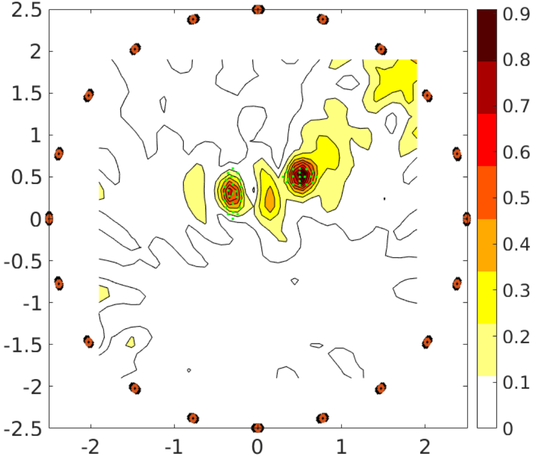
- More receivers leads to more stable reconstructions
- Theoretically, transmitter locations need to surround obstacles - experimentally we can use fewer
- Physical and financial constraints can be used to select measurement geometry

Quasi-backscattering measurements



- Measure scattered field only near transmitter location
- Easier data collection when obstacles are large (e.g., plane canopies and TechSat 21)

Simulated results



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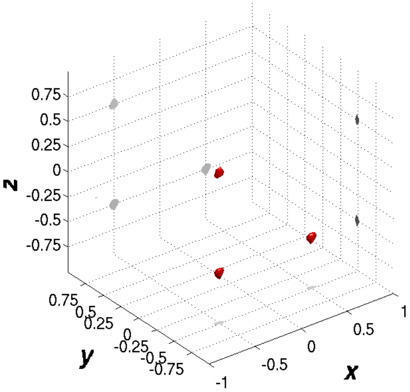

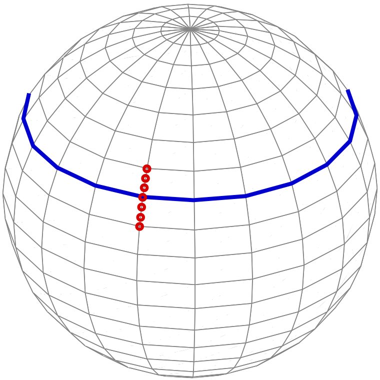

Extensions

- Large objects
  - Requires deeper mathematical theory, but idea is the same
- Limited aperture transmitter locations
  - Time domain data
  - Multi-frequency data
  - Couple with nonlinear optimization routines
- Different physical models
  - Similar methods have been shown to work for time dependent acoustic data and time harmonic elastic data
  - The necessary components (nearly linear scattering and knowledge of how waves propagate in free space) are available in many physical situations

Chapter Two

Uncertainty Propagation and Bayesian Inverse Problems

The problem with noisy measurements

For each $\zeta \in \mathcal{Z}$, we solve

$$\int_{\Gamma_i} u^s_y(x) g(y)\, \mathrm{d}s(y) = u^i_\zeta(x)$$

- Uncertainty in measurements of u^s
- Uncertainty in shape of u^i
- Uncertainty in location of transmitters and receivers
- Uncertainty in model (constant background parameters?)

How can we quantify this lack of knowledge in our reconstructions?

Monte Carlo approach

Seek the probability law of g_ζ and an estimate of its statistics.

Assume errors can be separated so that

$$\int_{\Gamma_i} u^s_y(x) g(y)\,\mathrm{d}s(y) = u^i_\zeta(x) + \epsilon.$$

A simple Monte Carlo-type method (or, e.g., spectral expansion method) can be used to find statistics of g_ζ.

Ignores modeling assumptions which validate reconstruction algorithm.

Bayesian Inverse Problems

$$\Delta u_y^s(x) + k^2(1 - m(x))u_y^s(x) = -k^2(1 - m(x))u_y^i(x)$$
$$\Delta u_y^i(x) + k^2 u_y^i(x) = 0$$
$$\lim_{|x|\to\infty} |x|^{1/2}\left(\frac{\partial u^s}{\partial \nu} - iku^s\right) = 0,$$

- Calculate probability law of m, given the data we collected
- Requires a priori information about how errors are distributed
- Bayesian approach helps to incorporate lack of information in a principled fashion

$$\boxed{\pi_{\text{post}}(m|u^s_{\text{obs}}) \propto \pi_{\text{like}}(u^s_{\text{obs}}|m)\pi_{\text{prior}}(m)}$$

Bayesian Inverse Problems

$$
\begin{aligned}
&\Delta u_y^s(x) + k^2(1 - m(x))u_y^s(x) = -k^2(1 - m(x))u_y^i(x) \\
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$$
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$$

Components needed for Bayesian inversion

- Function, $f(m)$, mapping from m to u^s_{obs}
  - Numerical approximation to scattering PDE
  - Assume
    $$u^s_{\text{obs}} = f(m) + \eta$$
    where $\eta \sim \mathcal{N}(0, \Gamma_{noise})$
  - Allows us to write
    $$\pi_{\text{like}}(u^s_{\text{obs}}|m) \propto \exp\left(-\frac{1}{2}(f(m) - u^s_{\text{obs}})^T \Gamma^{-1}_{\text{noise}}(f(m) - u^s_{\text{obs}})\right).$$
- A priori distribution for m
  - Assume normal distribution for simplicity,
    $$m \sim \mathcal{N}(m_0, \mathcal{C}_0)$$
  - For technical reasons, $\mathcal{C}_0 = \mathcal{A}^{-2}$ must be related to the inverse of a solution map for an elliptic PDE

Linearized case

Assume that errors are small. Then,

$$u_{\text{obs}}^{s} \approx f(\hat{m}) + F(m - \hat{m}) + \eta$$

is a good approximation where F is the Frechet derivative of f and

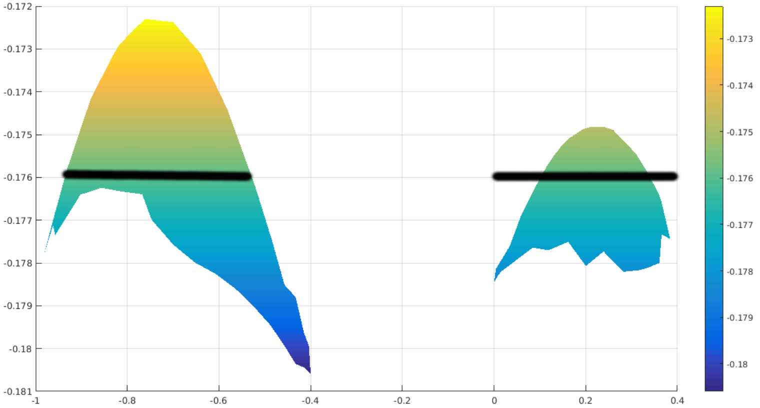
$$\hat{m} = \min \left(\| f(m) - u_{\text{obs}}^{s} \|^2 + \| \mathcal{A}(m - m_0) \|^2 \right) .$$

In this case,

$$\boxed{\pi_{\text{post}}(m | u_{\text{obs}}^{s}) \sim \mathcal{N} \left(\hat{m}, \left(F^{*} \Gamma_{\text{noise}}^{-1} F + \mathcal{C}_0^{-1} \right)^{-1} \right)}$$

Statistics on such a distribution and samples from it can be obtained easily (but maybe slowly).

Sample reconstructions



Estimate of mean

Extensions

- Non-linear assumptions
  - Harder (impossible?) to write closed-form solution
  - Monte Carlo-type sampling can be used, but requires many PDE solves
  - Lots of research on how to address these issues, but problems is not solved
- Speed of solution
  - PDE needs to be solved for each parameter (e.g., m on every point of a finite element grid)
  - Need fast solution techniques, particularly for non-linear problem
- A priori assumptions
  - Infinite dimensional Bayesian inverse problems use prior distribution to ensure that solution exists
  - We would like to use physical reasoning or bootstrap based on deterministic reconstructions

Answers to challenges

Can we design general techniques which are useful at all of these scales?

- Many problems have similar characteristics which lead to useful reconstruction techniques
- Better understanding of physical situation can be incorporated into model uncertainties

How expensive are these techniques?

- Cost of data collection can be reduced by better understanding model
- Fast reconstructions available when looking for limited data

Are we confident in our results?

- Different uncertainties can be included in reconstruction algorithms
- Speed of reconstructions is decreased, but we gain extra information

Thanks!