Percolation: Theory and Applications

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OUTLINE

- Introduction/Setup

- Basic Results

- Example of Application

Introduction

Original problem: Broadbent and Hammersley(1957)

Suppose a large porous rock is submerged under water for a long time, will the water reach the center of the stone?

Related problems:

How far from each other should trees in an orchard (forest) be planted in order to minimize the spread of blight (fire)?

How infectious does a strain of flu have to be to create a pandemic? What is the expected size of an outbreak?

Setup: 2D Bond Percolation

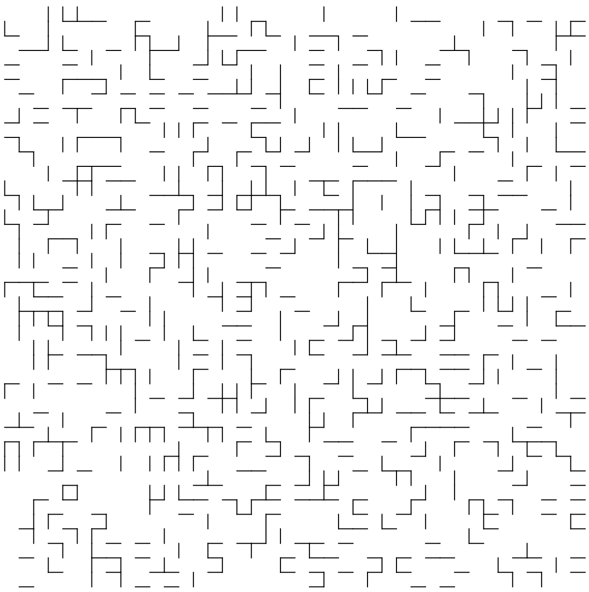
- Stone: a large two dimensional grid of channels (edges). Edges in the grid are *open* or present with probability p ($0 \leq p \leq 1$) and *closed* or absent with probability $1 - p$.

- Pores: open edges and p determines the porosity of the stone.

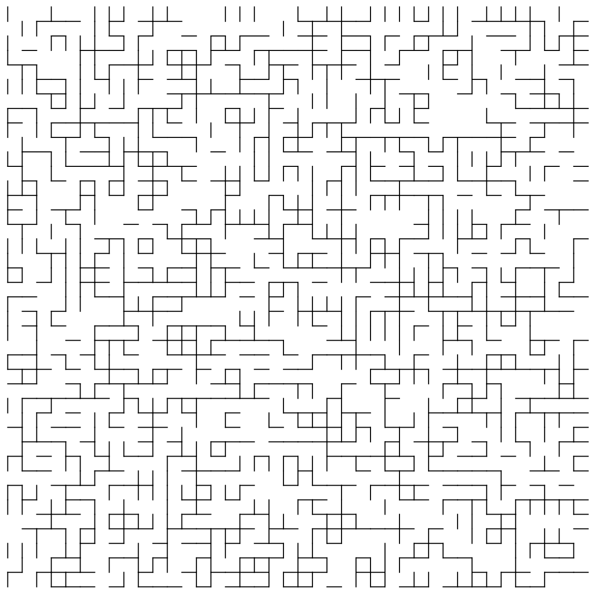
A contiguous component of the graph of open edges is called an *open cluster*. The water will reach the center of the stone if there is an open cluster joining its center with the periphery.

Similarly, in the orchard example, p is the probability that blight will spread to an adjacent tree and minimizing the spread corresponds to minimizing the size of the largest open cluster.

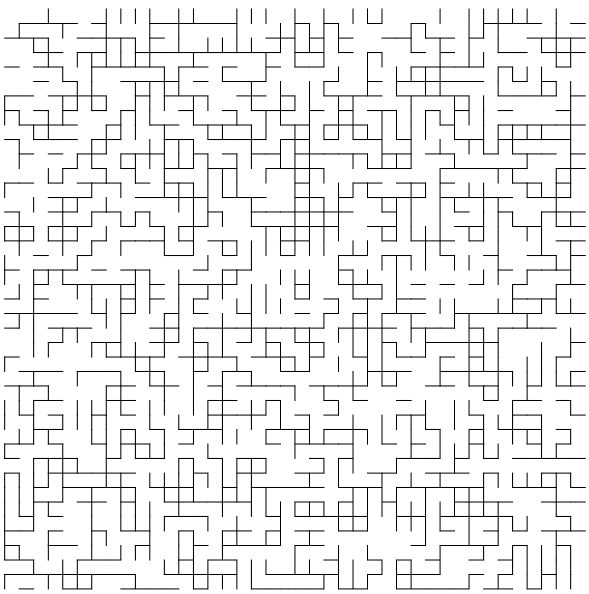
Setup: 2D Bond Percolation



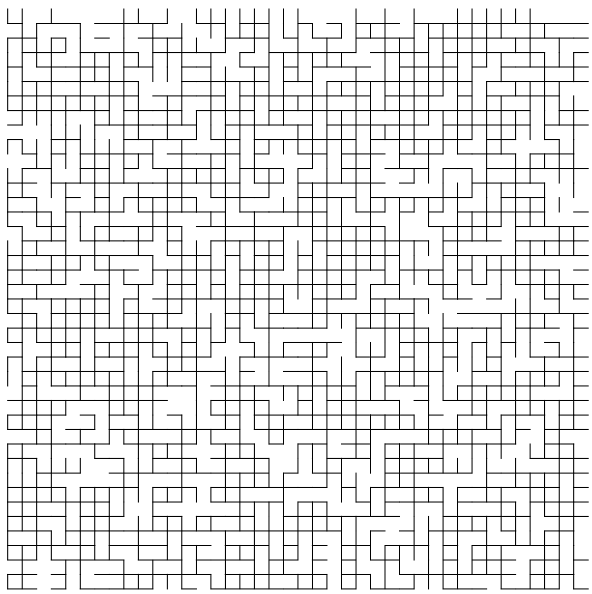
p=0.25



p=0.48



p=0.52



p=0.75

Setup: Bond Percolation

General Bond Percolation Model

- The space of the model is $\mathbf{Z}^n$ or any infinite graph.

- The edges are open or closed with probability p, which may depend on the properties of the edge (e.g. degree).

- Open cluster is a connected component of the open edge graph.

- The network is said to *percolate* if there is an infinite open cluster containing the origin.

If the graph is translation invariant there is no difference between the origin and any other vertex.

Setup: Site Percolation

Site Percolation Model

- The space of the model is $\mathbf{Z}^n$ or any infinite graph.
- The vertices are open or closed with probability p, which may depend on the properties of the vertex (e.g. degree).
- Open cluster is a connected component of the open vertex graph.
- The network is said to *percolate* if there is an infinite open cluster containing the origin.

Every bond percolation problem can be realized as a site percolation problem (on a different graph). The converse is not true.

Setup: Why Percolation?

- Percolation provides a very simple model of random media that nevertheless retains enough realism to make its predictions relevant in applications.

- It is a test ground for studying more complicated critical phenomena and a great source of intuition.

Basic Results: Quantities of Interest

- $|C|$ — the *size of the open cluster at 0*, where C stands for the open cluster itself;

- $\theta(p)$ — *percolation probability*, defined as

$$\theta(p) = P_p(|C| = \infty);$$

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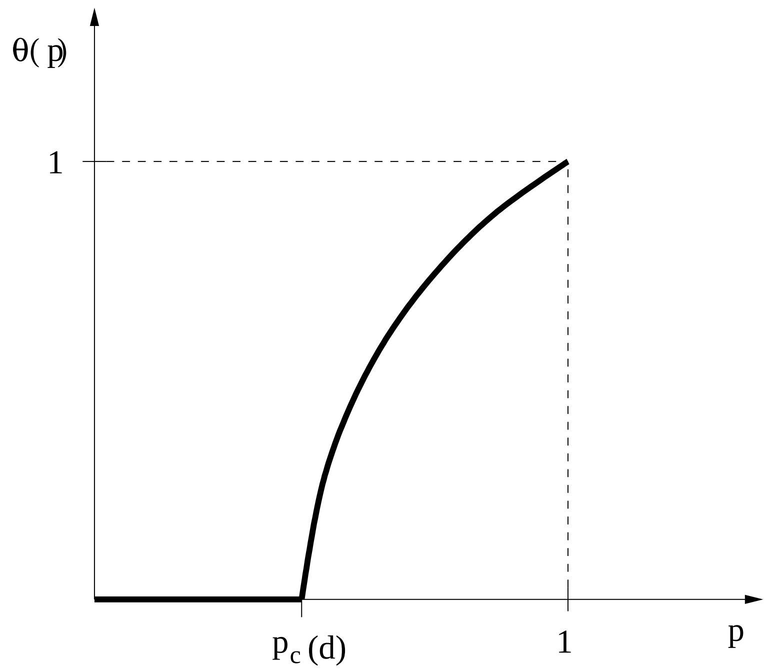
Basic Results: Percolation Probability

Exact shape of $\theta(p)$ is not known but it is believed to be a continuous function of p



Percolation thus has three distinct phases

1) *subcritical* if $p < p_c$
2) *critical* if $p = p_c$
3) *supercritical* if $p > p_c$

Basic Results: Quantities of Interest

- $|C|$ — the *size of the open cluster at 0*, where C stands for the open cluster itself;

- $\theta(p)$ — *percolation probability*, defined as

$$\theta(p) = P_p(|C| = \infty);$$

- $p_c(d)$ — *critical probability*, defined as

$$p_c(d) = \sup\{p : \theta(p) = 0\};$$

-

Basic Results: Critical Probability

Theorem. *If* $d \geq 2$ *then* $0 < p_c(d) < 1$.

The exact value of $p_c(d)$ is known only for a few special cases:

- $p_c^{\text{bond}}(1) = p_c^{\text{site}}(1) = 1$
- $p_c^{\text{bond}}(2) = 1/2$, $p_c^{\text{site}}(2) \approx .59$
- $p_c^{\text{bond}}(\text{triangular lattice}) = 2\sin(\pi/18)$
- $p_c^{\text{bond}}(\text{hexagonal lattice}) = 1 - 2\sin(\pi/18)$

Theorem. *Probability that an infinite open cluster exists is 0 if* $p < p_c(d)$ *and 1 if* $p > p_c(d)$.

It is known that no infinite open cluster exists for $p = p_c(d)$ if $d = 2$ or $d \geq 19$.

Basic Results: Critical Probability

Some bounds on the critical probability are known

Theorem. *If G is an infinite connected graph and maximum vertex degree $\Delta < \infty$. The critical probabilities of G satisfy*

$$\frac{1}{\Delta - 1} \leq p_c^{\text{bond}} \leq p_c^{\text{site}} \leq 1 - (1 - p_c^{\text{bond}})^{\Delta}.$$

In particular, $p_c^{\text{bond}} \leq p_c^{\text{site}}$ and strict inequality holds for a broad family of graphs.

Basic Results: Quantities of Interest

- $|C|$ — the *size of the open cluster at 0*, where C stands for the open cluster itself;

- $\theta(p)$ — *percolation probability*, defined as

$$\theta(p) = P_p(|C| = \infty);$$

- $p_c(d)$ — *critical probability*, defined as

$$p_c(d) = \sup\{p : \theta(p) = 0\};$$

- $\chi(p)$ — the *mean size of the open cluster at the origin*, defined as

$$\chi(p) = E_p[|C|];$$

- $\chi^f(p)$ — the *mean size of the finite open cluster at the origin*, defined as

$$\chi^f(p) = E_p[|C| : |C| < \infty];$$

Basic Results: Subcritical Phase

If $p < p_c$ all open clusters are finite with probability 1.

Theorem. *Probability of a cluster of size n at 0 decreases exponentially with n. More precisely, there exists $\alpha(p) > 0$, $\alpha(p) \to \infty$ as $p \to 0$ and $\alpha(p_c) = 0$ such that*

$$P_p(|C| = n) \approx e^{-n\alpha(p)} \text{ as } n \to \infty$$

This also implies that $\chi(p)$ is finite for all p in the subcritical region.

Theorem. *Probability distribution for cluster radii decays exponentially with the radius, i.e.*

$$P_p(0 \leftrightarrow \partial B(r)) \approx e^{-r/\xi(p)}$$

where $\xi(p)$ — the characteristic length of exponential decay — is the mean cluster radius.

Basic Results: Supercritical Phase

If $p > p_c$, with probability 1 at least one infinite open cluster exists.

Theorem. *The infinite open cluster is unique with probability 1.*

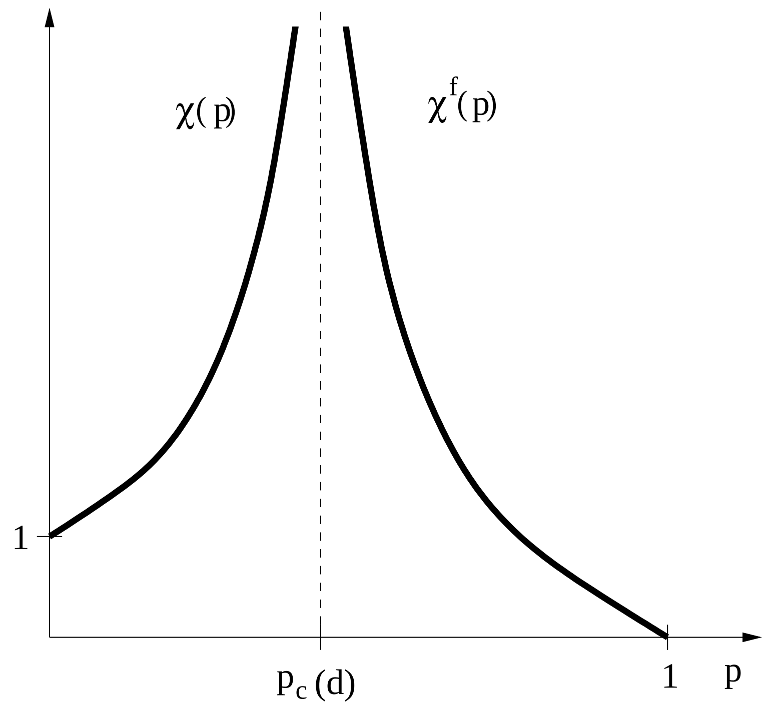
Theorem. *Probability of a finite open cluster of size n at 0 decreases exponentially with n. More precisely, there exist functions $\beta_1(p)$ and $\beta_2(p)$, satisfying $0 < \beta_2(p) \leq \beta_1(p) < \infty$, such that*

$$\begin{aligned} \exp(-\beta_1(p) n^{(d-1)/d}) &\leq P_p(|C| = n) \\ &\leq \exp(-\beta_2(p) n^{(d-1)/d}) \end{aligned}$$

Because $\chi(p)$ is infinite for $p > p_c$ the truncated mean — $\chi^f(p)$ — over finite clusters only is considered.

Basic Results: $\chi(p)$

The general shape of $\chi(p)$ is believed to be as follows



Application: Network Robustness and Fragility

Problem: How many random nodes can be removed before a network looses connectivity? How many of the highly connected nodes can be removed before the network looses connectivity?

Use site percolation model on a random graph with a given degree distribution p_k and vertex occupation probability q_k depending on the vertex degree.

Allowing q_k to vary with k allows to study various types of attacks: random if $q_k = q$ is independent of k, targeted deletion of high degree nodes if $q_k = H(k_{max} - k)$.

Application: Network Robustness and Fragility

Using formalism of generating functions it can be shown that the generating function H_0 of cluster size $|C|$ at a random vertex satisfies

$$
\begin{aligned}
H_0(x) &= 1 - F_0(1) + xF_0(H_1(x)) \\
H_1(x) &= 1 - \frac{1}{z}(F_0'(1) + xF_0'(H_1(x))) \\
F_0(x) &= \sum_{k=0}^{\infty} p_k q_k x^k
\end{aligned}
$$

and z is the mean graph degree, and

$$\chi(q) = H_0'(1)$$

Application: Network Robustness and Fragility

Although closed form solutions to the above equations do not exist in general, it is possible to compute H_0 to any degree of accuracy by iterating equations for H_1 and then substituting into the equation for H_0.

In the case $q_k = q$ (uniform distribution) it can be shown that $\chi(q)$ diverges at

$$q_c = \frac{1}{G''(1)}$$

where $G = \frac{1}{z}\sum_k p_k x^k$. This is the percolation threshold probability.

Application: Network Robustness and Fragility

$$p_k = \begin{cases} 0 & \text{if } k = 0 \\ Ck^{-\tau} e^{-k/\kappa} & \text{if } k \geq 1 \end{cases}$$

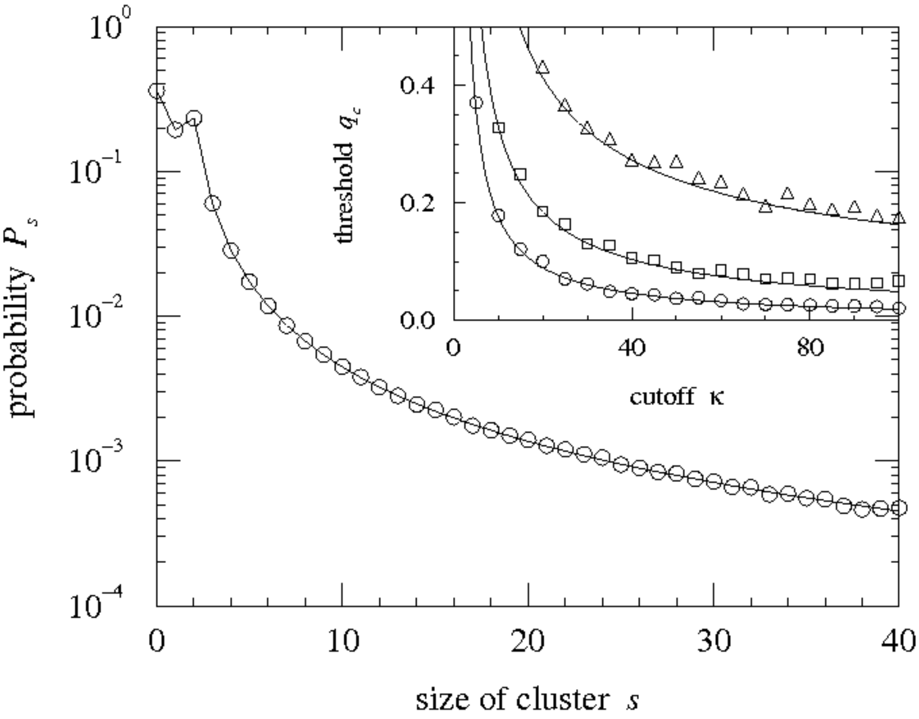


FIG. 1. Probability P_s that a randomly chosen vertex belongs to a cluster of s sites for $\kappa = 10$, $\tau = 2.5$, and $p = 0.65$ from numerical simulation on systems of 10^7 sites (circles) and our exact solution (solid line). Inset: the percolation threshold q_c from Eq. (12) (solid lines), versus computer simulations with $\tau = 1.5$ (circles), 2.0 (squares), and 2.5 (triangles).

Application: Network Robustness and Fragility

If the highest degree vertices are removed first, $q_k = H(k_{\text{max}} - k)$, the probability that a random vertex does not belong to the giant open cluster is

$$S = 1 - H_0(1) = F_0(1) - F_0(u)$$

where u solves

$$u = 1 - \frac{1}{z}(F'(1) + F'(u))$$

These equations can be solved numerically.

Application: Network Robustness and Fragility

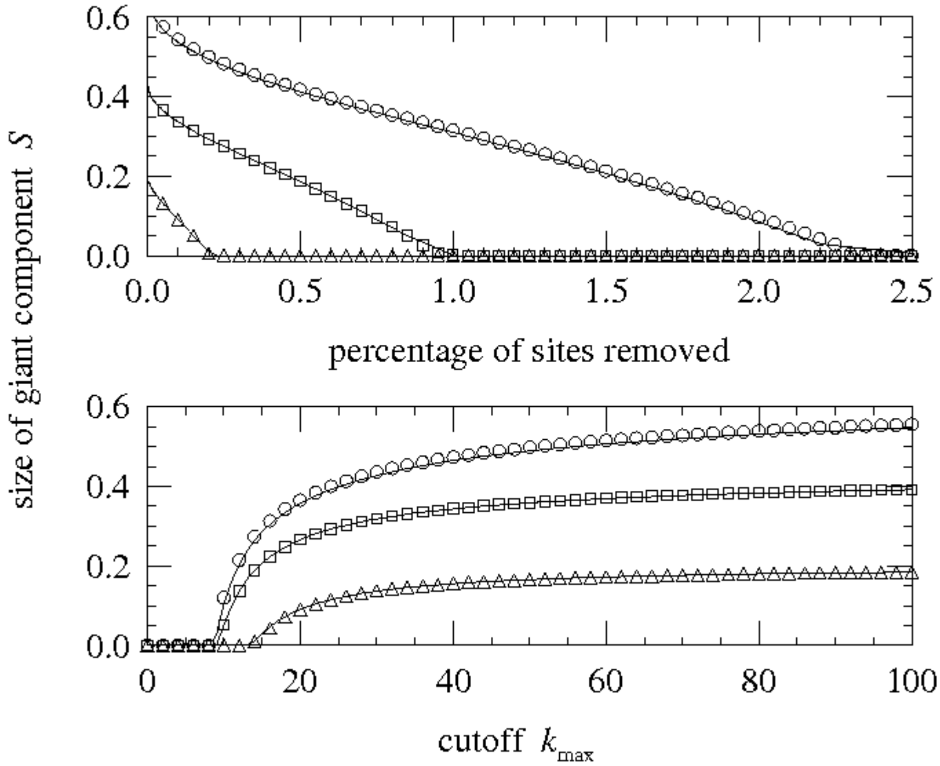


FIG. 2. Size of the giant component S in graphs with power-law degree distribution and all vertices with degree greater than $k_{\max}$ unoccupied, for $\tau = 2.4$ (circles), 2.7 (squares), and 3.0 (triangles). Points are simulation results for systems with 10^7 vertices, solid lines are the exact solution. Upper frame: as a function of fraction of vertices unoccupied. Lower frame: as a function of the cutoff parameter $k_{\max}$.

Bibliography

Reka Albert and Albert Laszlo Barabasi, Statistical mechanics of complex networks, Reviews of modern Physics, 74, Jan. 2002.

Duncan Callaway, M. E. J. Newman, Steven H. Strogatz, and Duncan J. Watts, Network robustness and fragility: Percolation on random graphs, arXiv:cond-mat/0007300, Oct. 2000.

Geoffrey Grimmet, Percolation, Grundlehren der mathematischen Wissenschaft, vol 321, Springer, 1999.

M. E. J. Newman, The structure and function of complex networks, arXiv:cond-mat/0303516v, Mar. 2003.