Area Laws for Entanglement

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joint work with
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Quantum Information Theory

Goal: Lay down the theory for future quantum-based technology (quantum computers, quantum cryptography, ...)

Quant. Comm.
Entanglement theory
Q. error correc. + FT
Quantum comp.
Quantum complex. theo.

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Ultimate limits to information transmission

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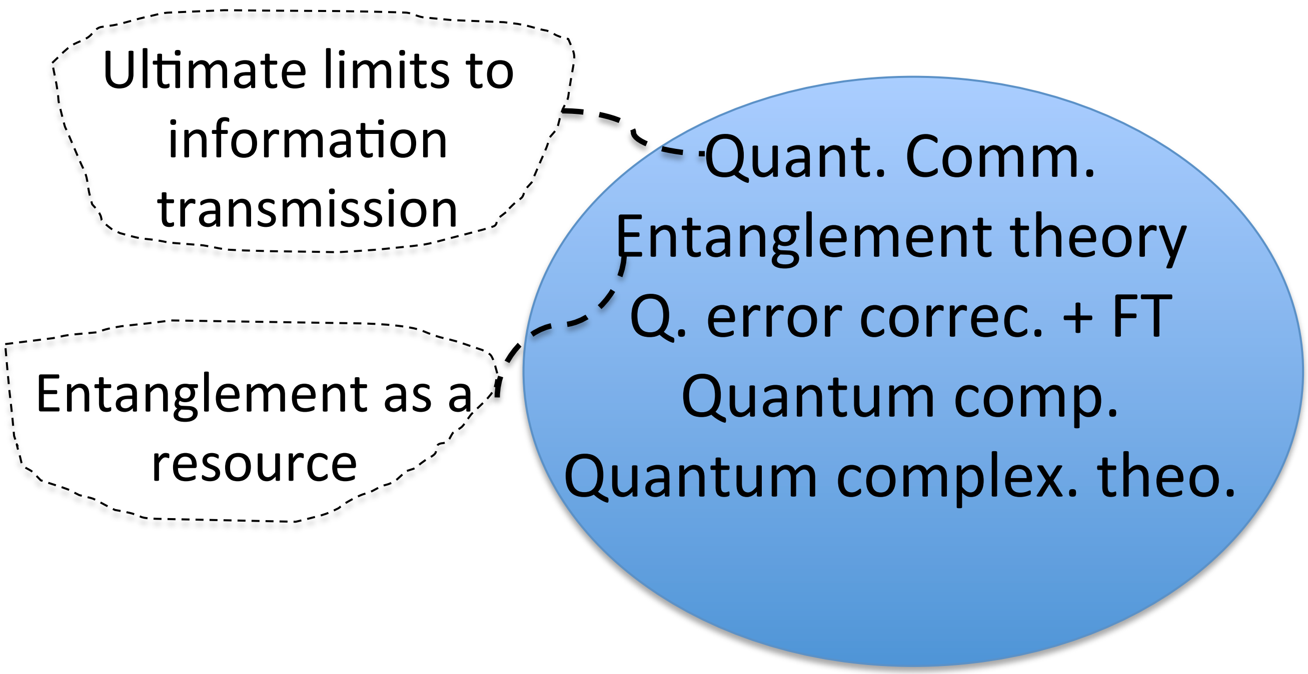

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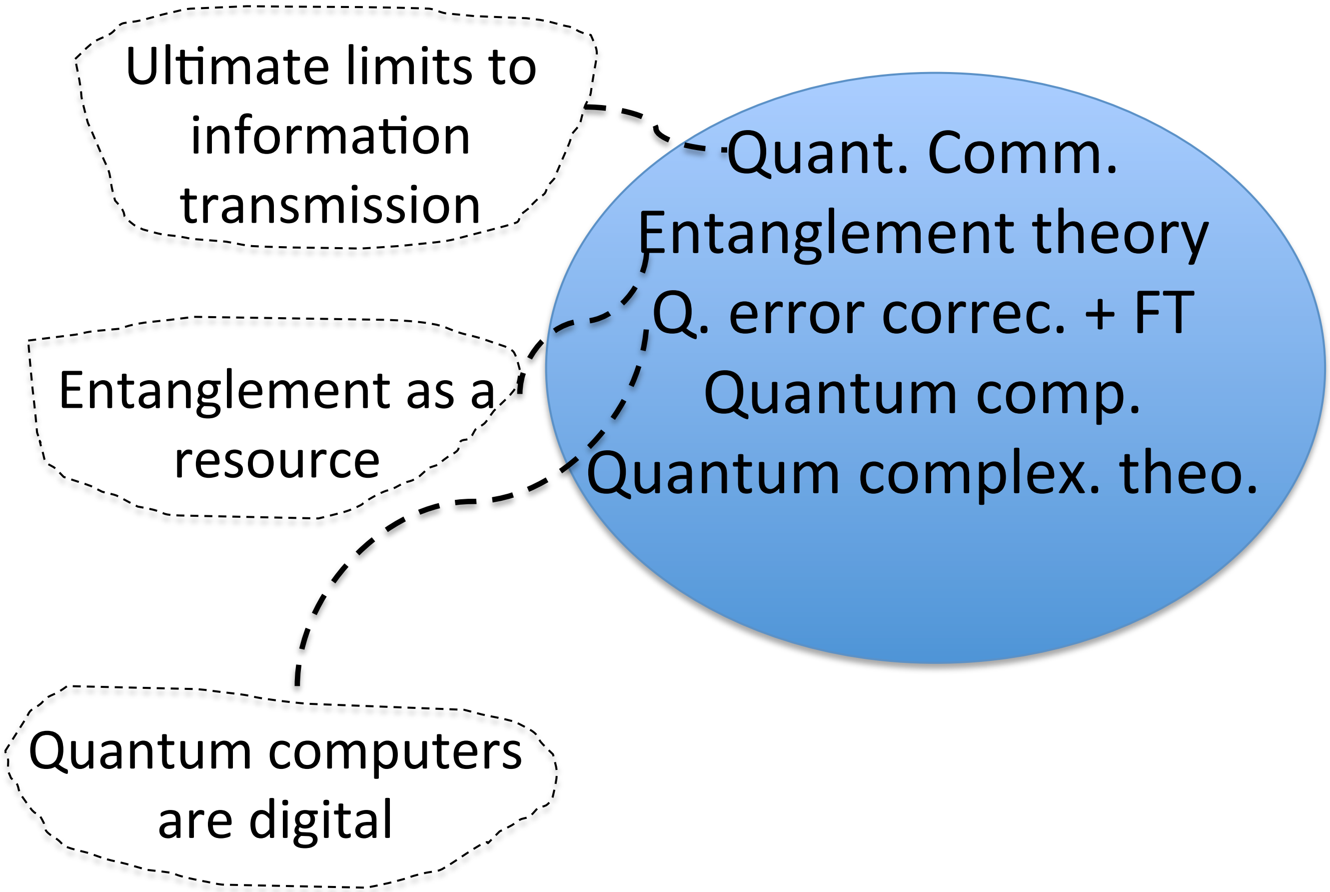

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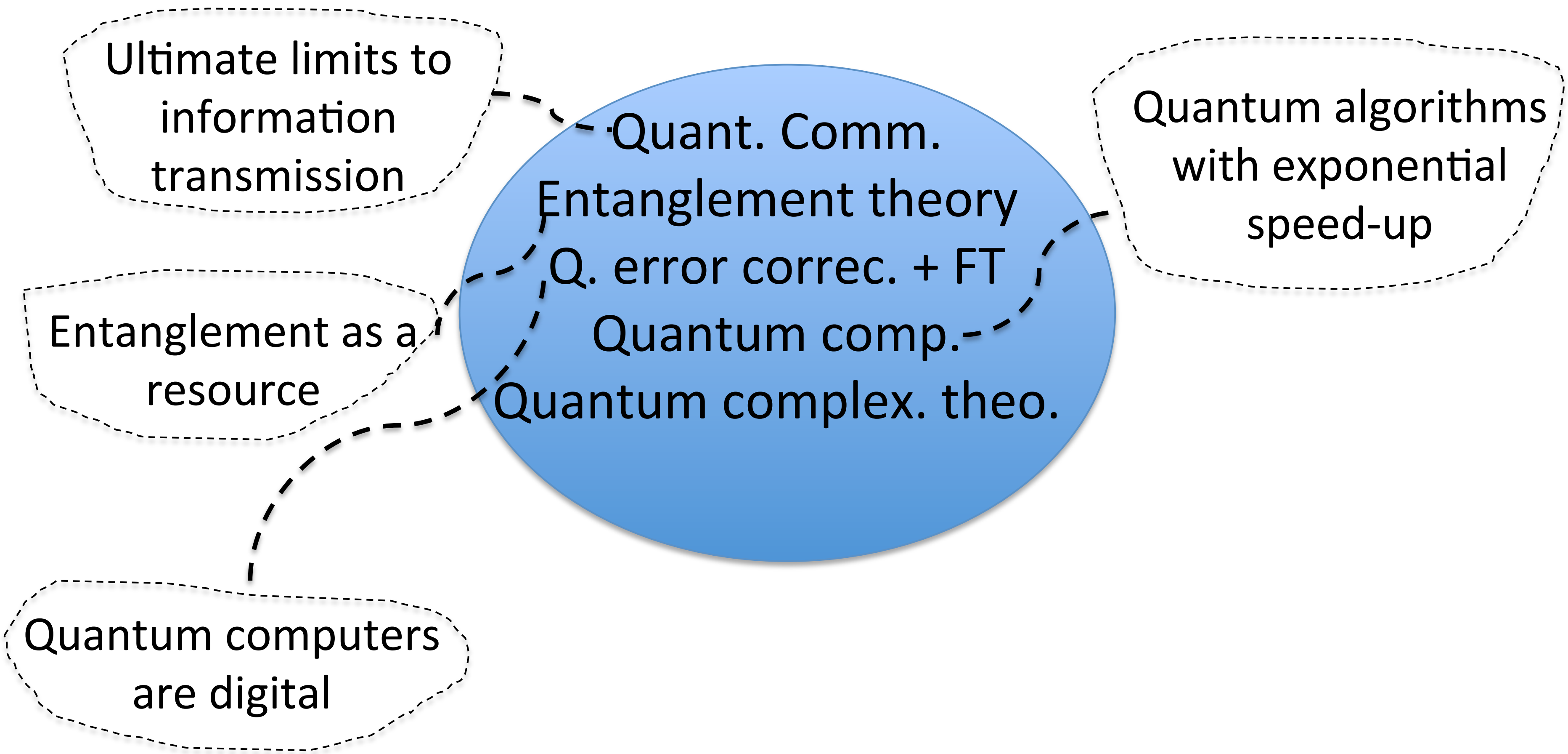

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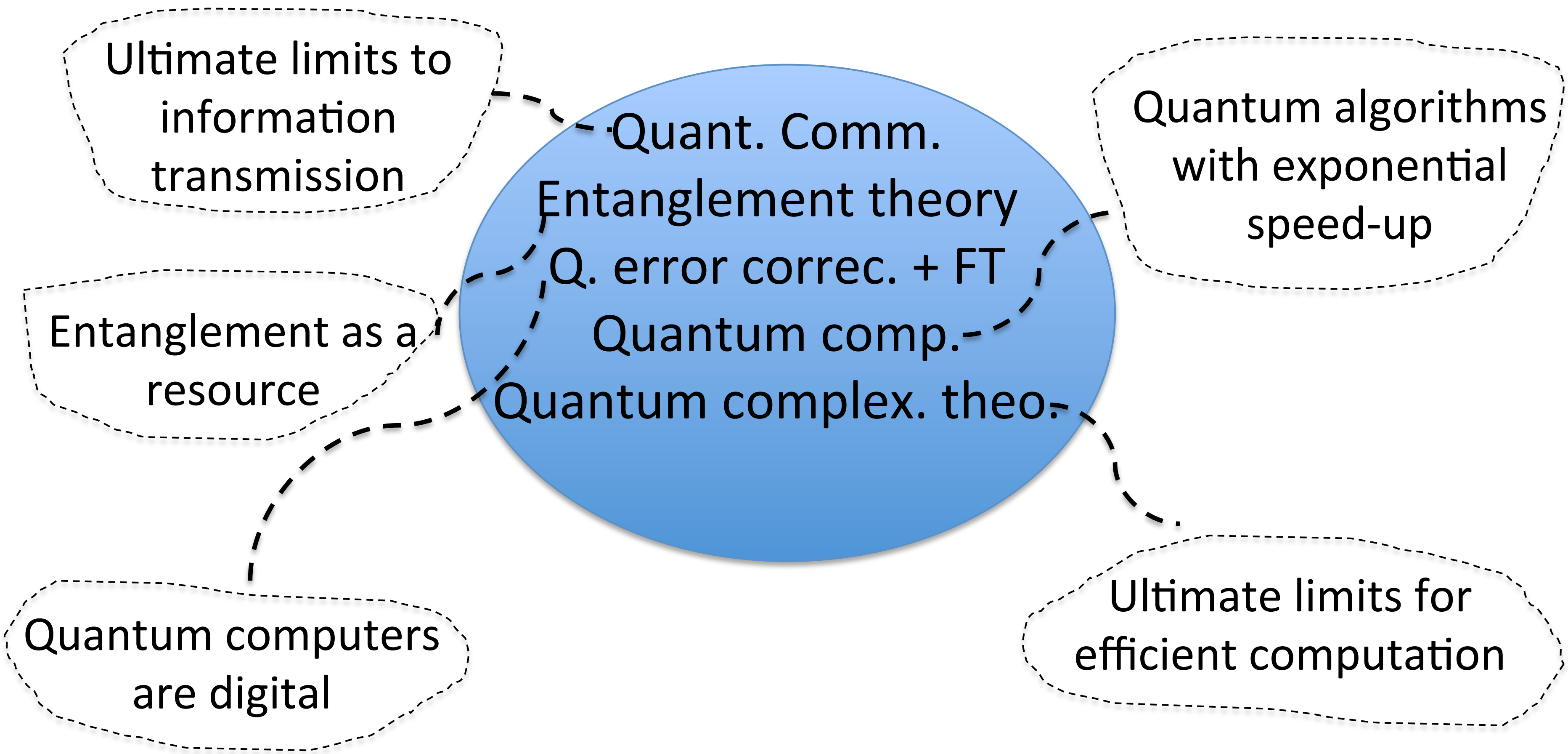

QIT Connections

QIT

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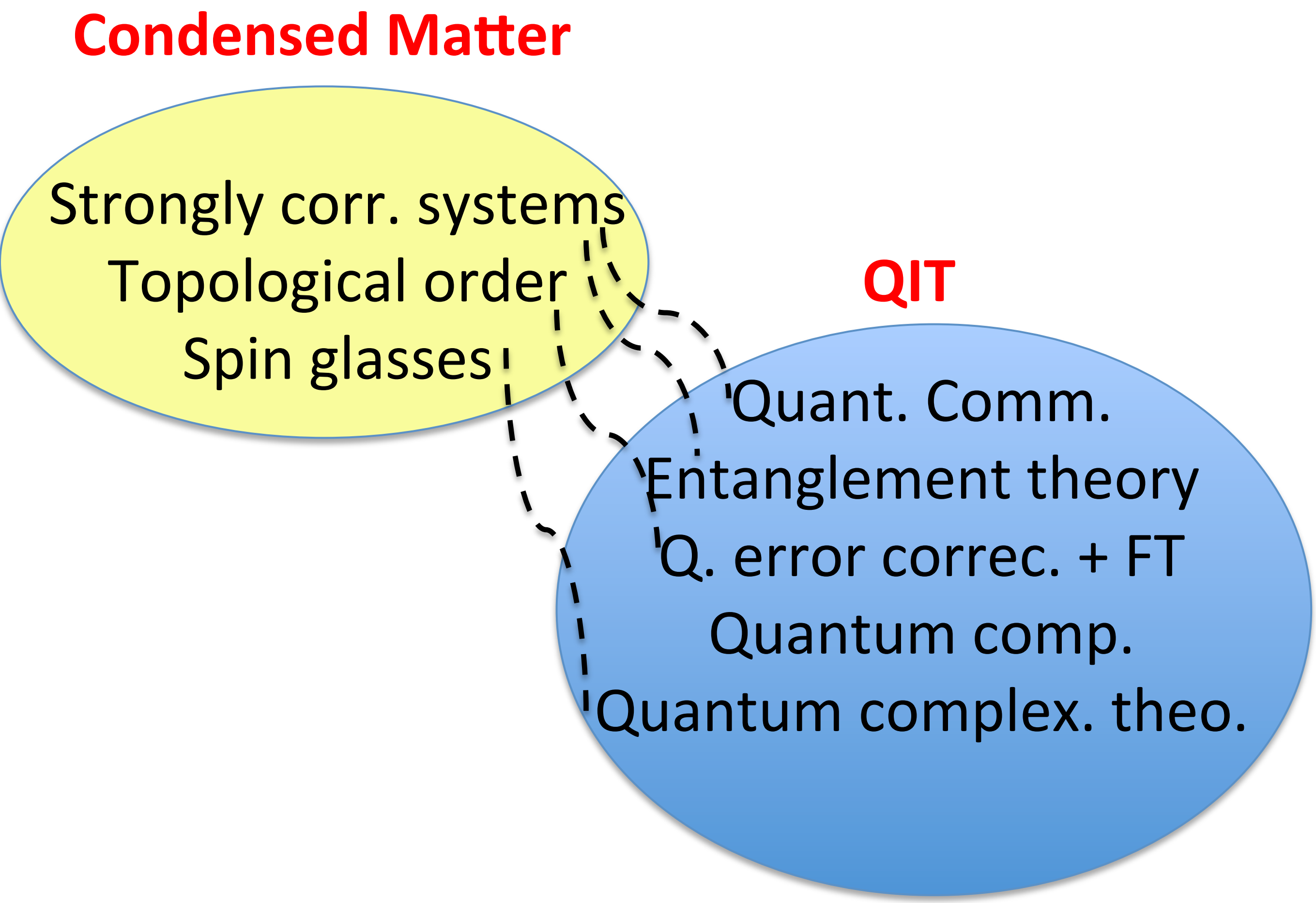

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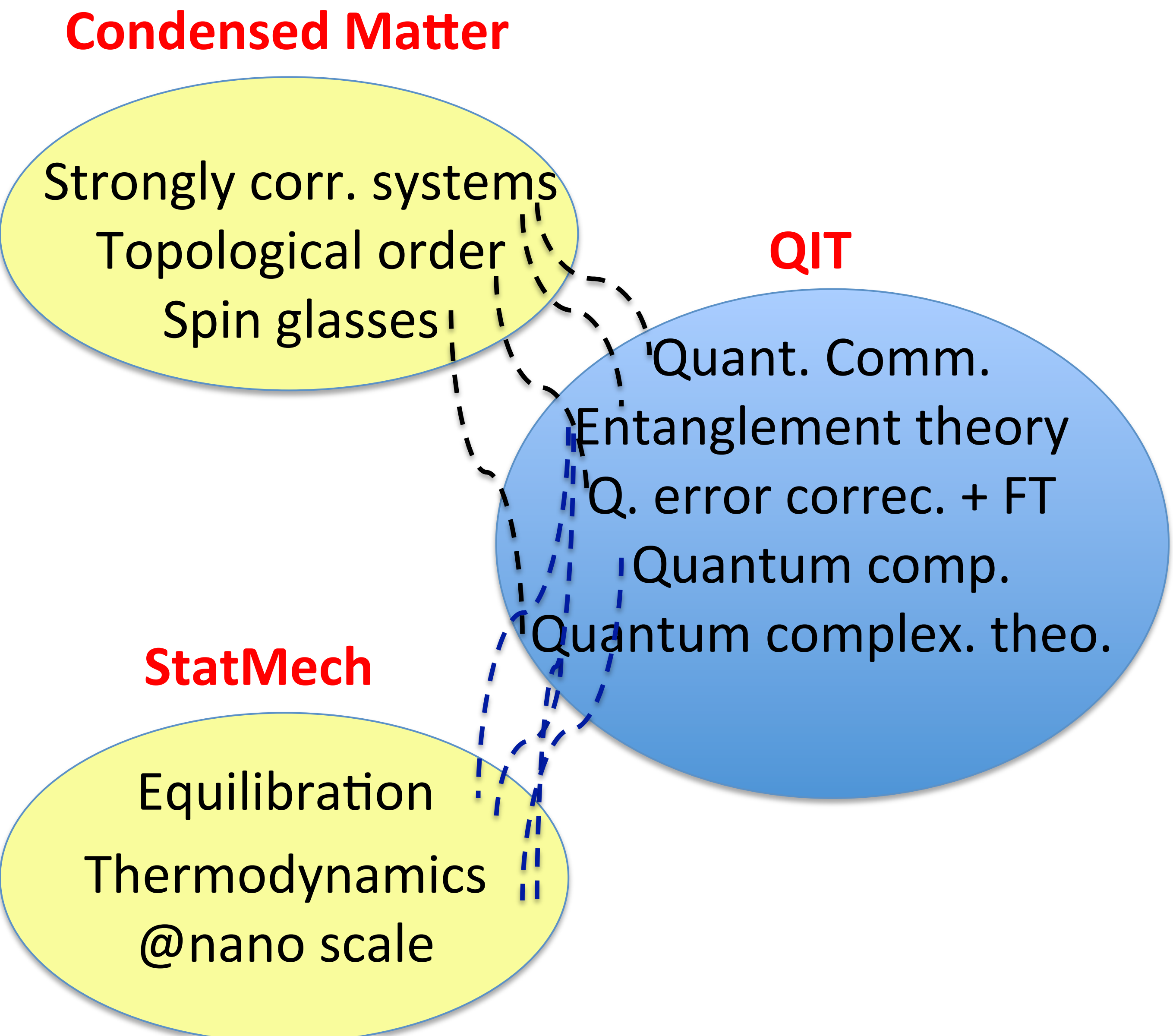

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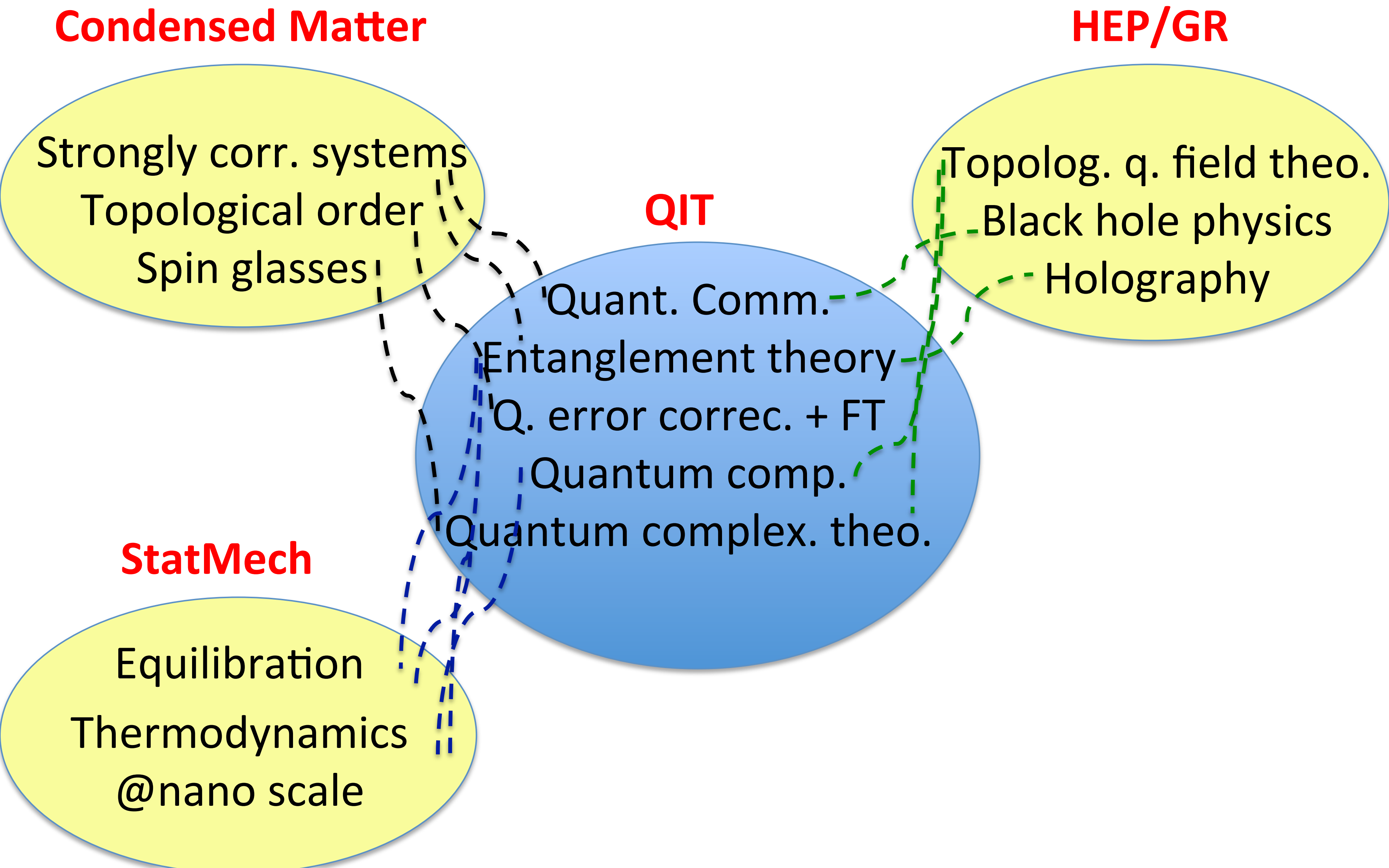

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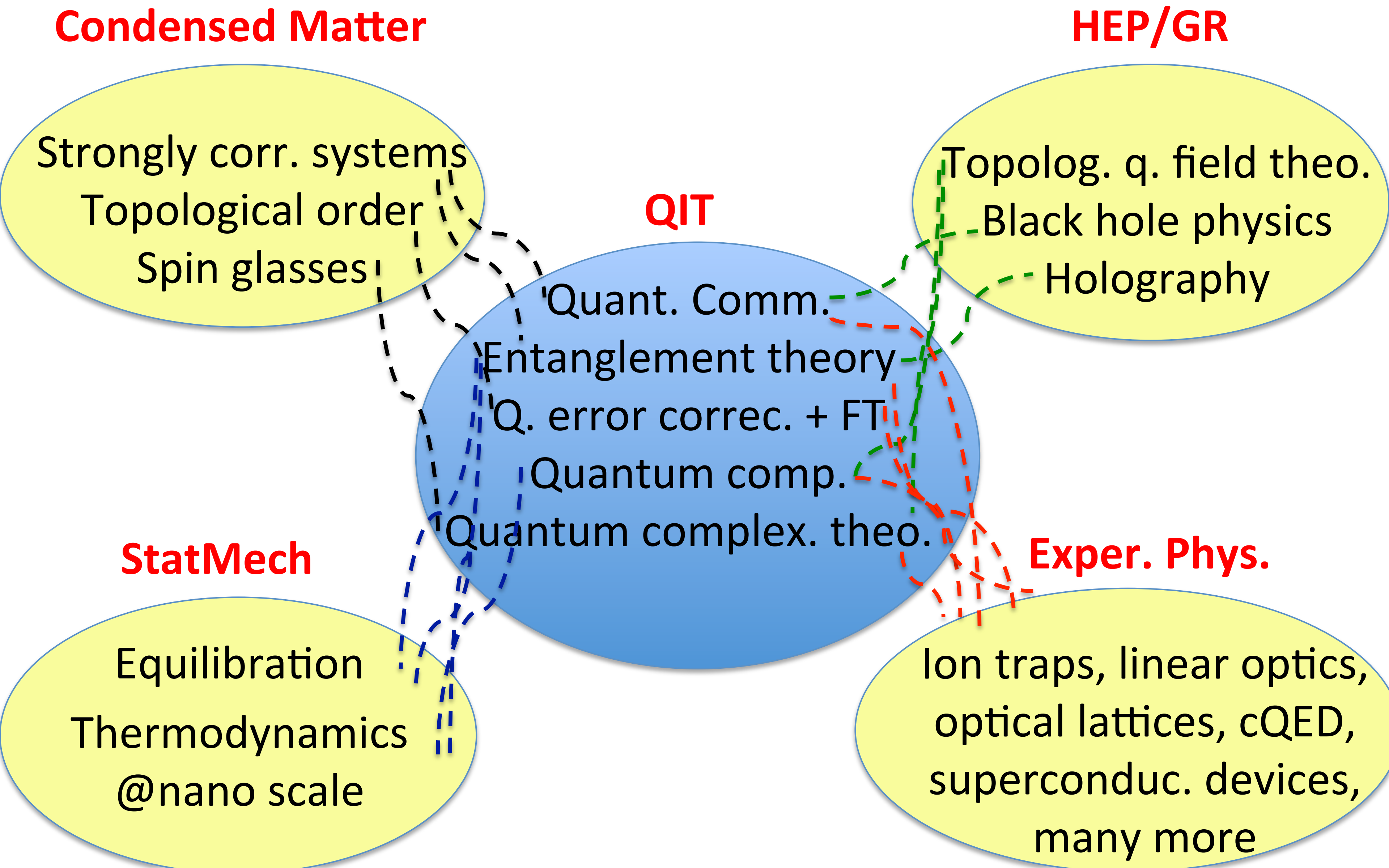


This Talk

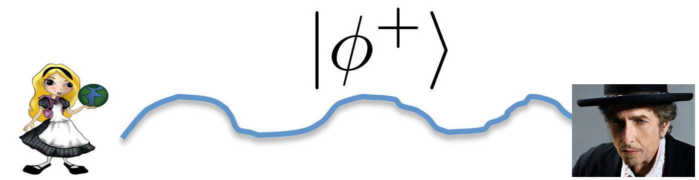
Goal: give an example of these emerging connections:

Connect behavior of correlation functions to *entanglement*

Entanglement

Entanglement in quantum information science is a **resource** (teleportation, quantum key distribution, metrology, …)

Ex. EPR pair $|\phi^+\rangle = (|0,0\rangle + |1,1\rangle)/\sqrt{2}$



How to quantify it?

Bipartite Pure State Entanglement

Given $|\psi\rangle_{AB}$, its entropy of entanglement is

$$E(|\psi\rangle_{AB}) := S(\rho_A) = S(\rho_B)$$

Reduced State: $\rho_A := \mathrm{tr}_B(|\psi\rangle\langle\psi|_{AB})$

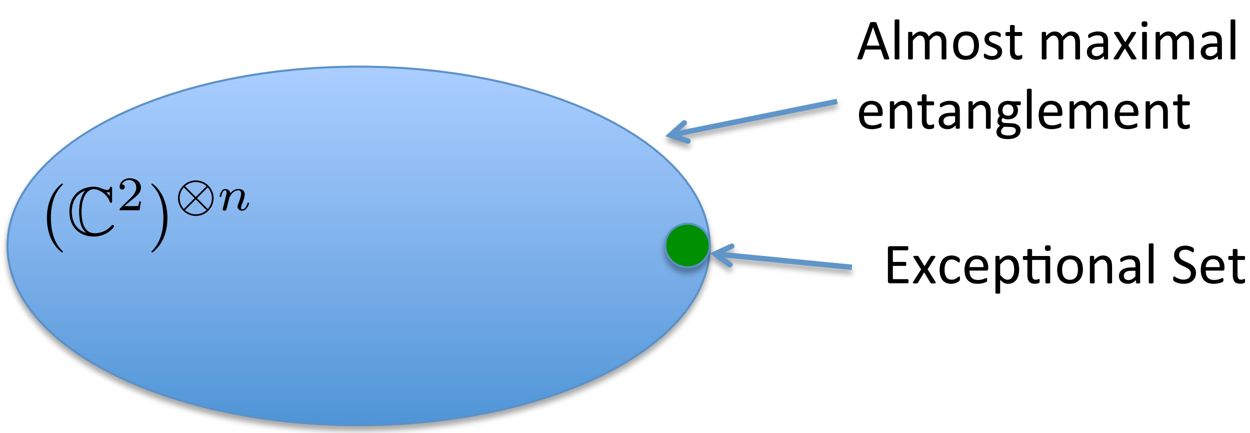
Entropy: $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ (Renyi Entropies: $S_\alpha(\rho) := \frac{1}{1-\alpha} \log \mathrm{tr}(\rho^\alpha)$)

Entanglement in Many-Body Systems

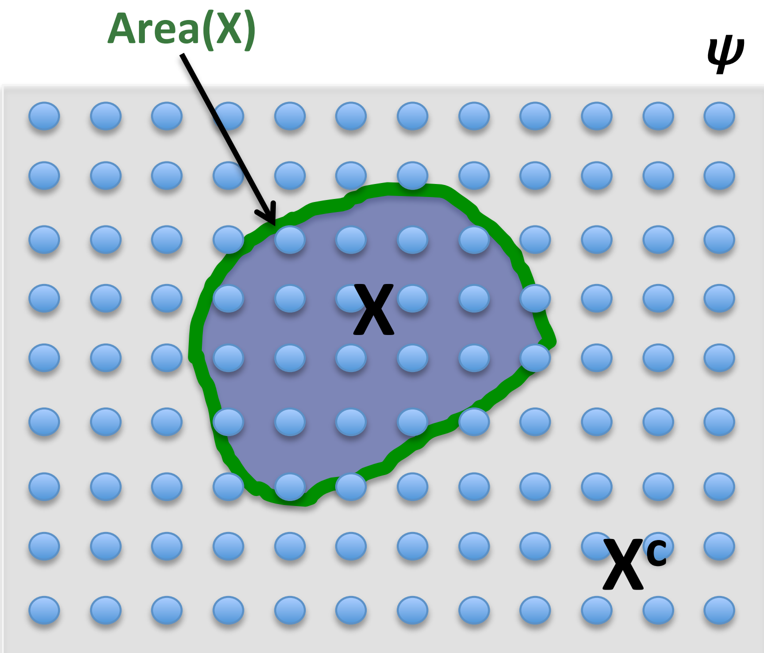
A quantum state ψ of n qubits is a vector in $(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle$$

For almost every state ψ, $S(X)_\psi \approx |X|$ (for any X with $|X| < n/2$)
$|X| :=$ # qubits in X



Area Law

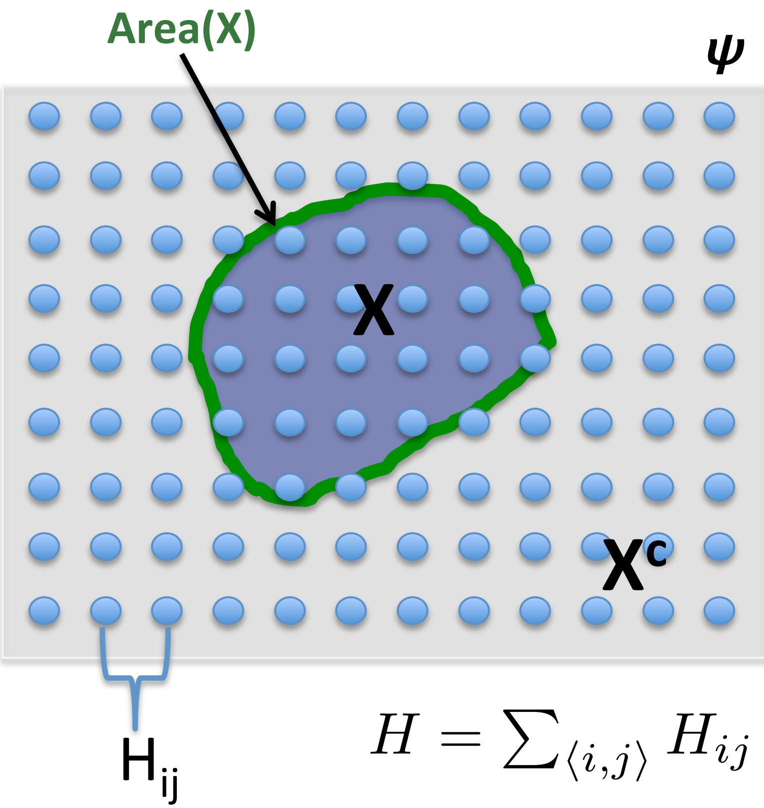


Def: ψ satisfies an area law if there is c > 0 s.t. for every region X,

$$S(X) \leq c \, \text{Area}(X)$$

Entanglement is Holographic

Area Law



$H = \sum_{\langle i,j \rangle} H_{ij}$

Def: ψ satisfies an area law if there is c > 0 s.t. for every region X,

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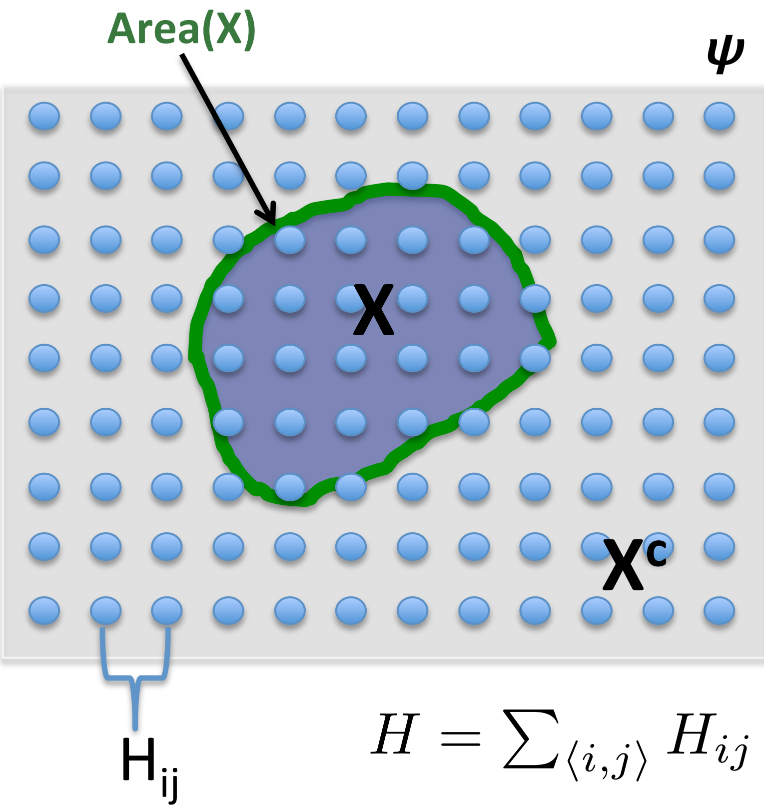
Entanglement is Holographic

When do we expect an area law?

Low-energy states of many-body local models: $H|\psi_0\rangle = E_0|\psi_0\rangle$

Area Law



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Low-energy states of many-body local models: $H|\psi_0\rangle = E_0|\psi_0\rangle$

(Bombeli *et al* ’86) massless free scalar field (connection to Bekenstein-Hawking entropy)

(Vidal *et al* ‘03; Plenio *et al* ’05, …) XY model, quasi-free bosonic and fermionic models, …

(Holzhey *et al* ‘94; Calabrese, Cardy ‘04) critical systems described by CFT (log correction)

⋮

(Aharonov *et al* ‘09; Irani ‘10) 1D model with *volume* scaling of entanglement entropy!

Why is Area Law Interesting?

- Connection to **Holography**.
- Interesting to study entanglement in physical states with an eye on **quantum information processing**.
- Area law appears to be connected to our ability to write-down **simple Ansatzes** for the quantum state.
  (e.g. tensor-network states: PEPS, MERA)

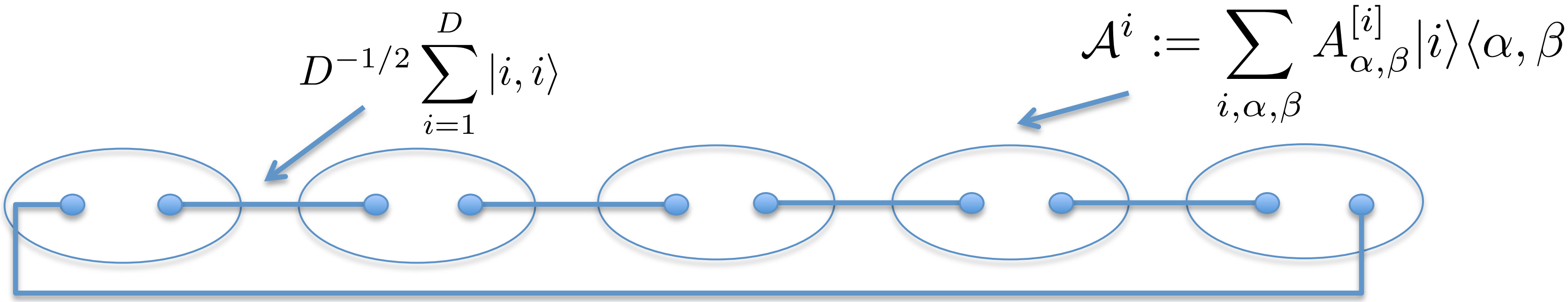
  This is known rigorously in 1D:

Matrix Product States

(Fannes, Nachtergaele, Werner '92; Affleck, Kennedy, Lieb, Tasaki '87)

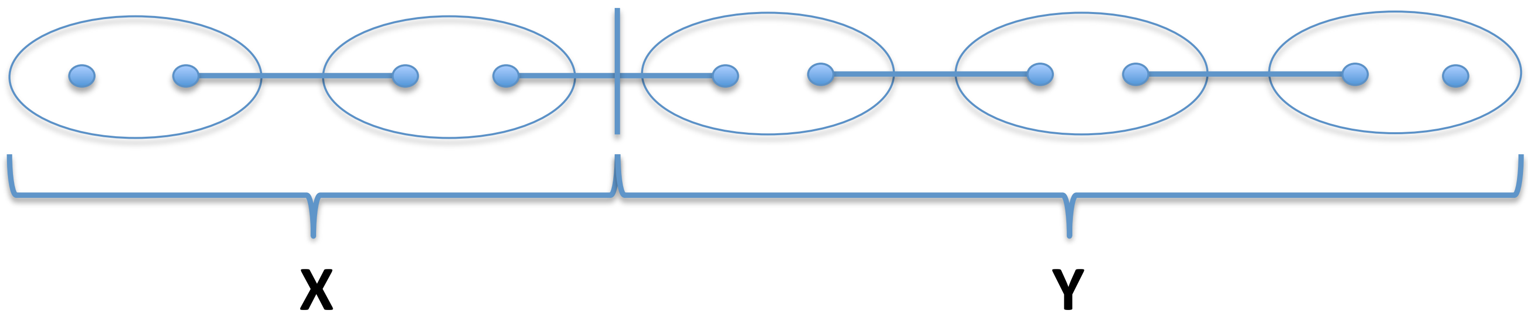
$$|\psi\rangle_{1,\ldots,n} = \sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2} tr\left(A_{i_1}^{[1]}\ldots A_{i_n}^{[n]}\right)|i_1,\ldots,i_n\rangle,\ \ A_j^{[l]} \in Mat(D,D)$$

D : bond dimension



- Only **nD²** parameters.
- Local expectation values computed in **nD³** time
- Variational class of states for powerful **DMRG** (White ')
- Generalization of product states (MPS with **D=1**)

MPS ⟺ Area Law



- For MPS, $S(\rho_X) \leq \log(D)$

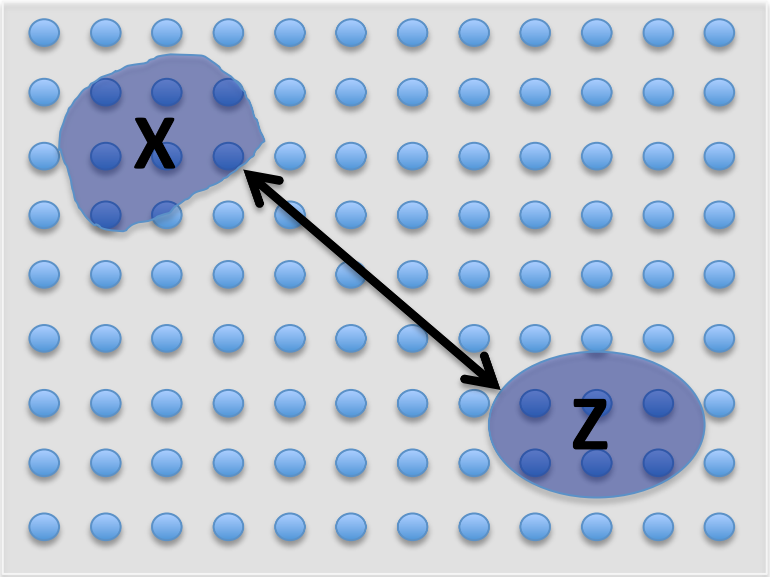
- (Vidal '03; Verstraete, Cirac '05)

  If ψ satisfies $S(\rho_X) \leq \log(D)$ for all X,
  then it has a MPS description of bond dim. D

  (obs: must use Renyi entropies)

Correlation Length

ψ



Correlation Function:

$$cor(X:Z)_\psi := \max_{M \in X, N \in Y} \langle\psi|M \otimes N|\psi\rangle - \langle\psi|M|\psi\rangle\langle\psi|N|\psi\rangle$$

Correlation Length: ψ has correlation length ξ if for every regions X, Z:

$$cor(X : Z)_\psi \leq 2^{-\,dist(X,\,Z)\,/\,\xi}$$

$$\mathrm{dist}(X, Z) := \min_{x \in X, z \in Z} \mathrm{dist}(x, z)$$

When there is a finite correlation length?

(Hastings '04) In any dim at zero temperature for *gapped* models
(for groundstates; $\xi = O(1/gap)$)

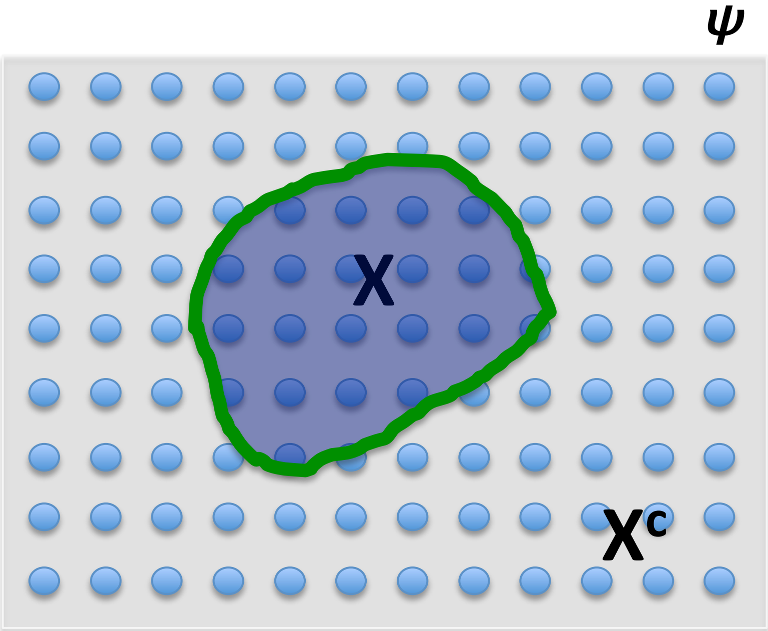
(Hastings '11; Hamza *et al* '12; …) In any dim for models with mobility gap
(many-body localization)

(Araki '69) In 1D at any finite temperature *T*
(for $\rho = e^{-H/T}/Z$; $\xi = O(1/T)$)

(Kliesch *et al* '13) In any dim at large enough *T*

(Kastoryano *et al* '12) Steady-state of fast converging dissipative
processes (e.g. *gapped* Liovillians)

Area Law from Correlation Length?



$$\mathrm{Ent}(X : X^c) \approx \sum_{i \in X^c} \mathrm{Ent}(X : X_i^c)$$

$$\approx \textstyle\sum_{i \in X^c} 2^{-\mathrm{dist}(X,i)/\xi}$$

$$= O(\mathrm{Area}(X)\xi)$$

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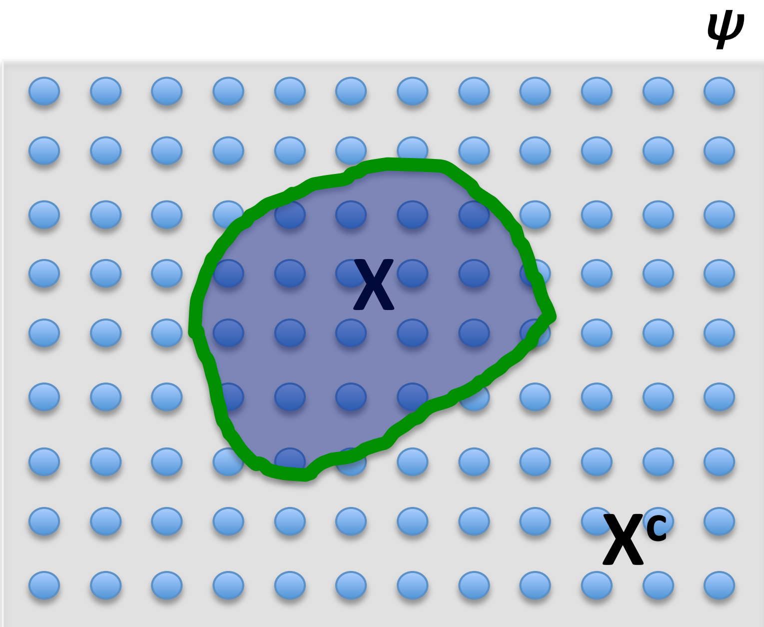

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$$\approx \sum_{i \in X^c} 2^{-\text{dist}(X,i)/\xi}$$
$$= O(\text{Area}(X)\xi)$$

That's incorrect!

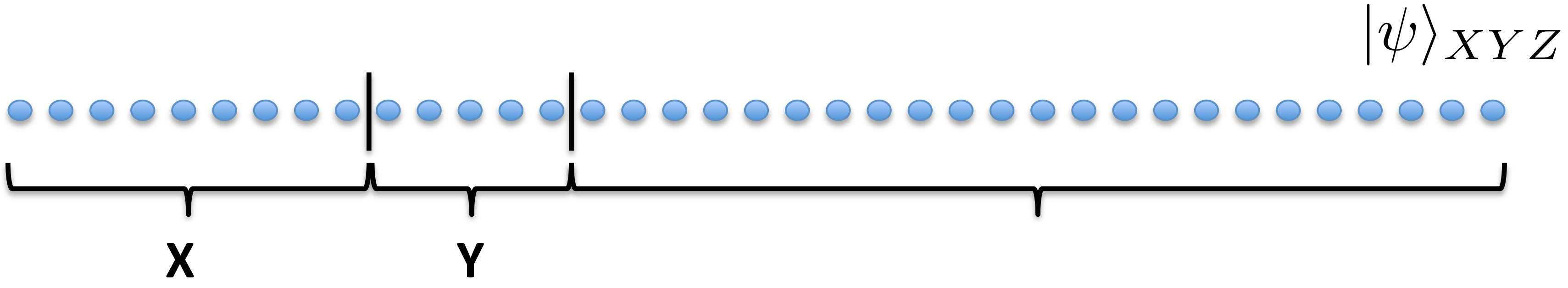
Ex. For almost every *n* qubit state: $S(X) \approx \text{vol}(X)$

and for all *i* in X^c, $\|\rho_{X X_i^c} - \rho_X \otimes \rho_{X_i}^c\|_1 \leq 2^{-cn}$

Entanglement can be *scrambled*, non-locally encoded

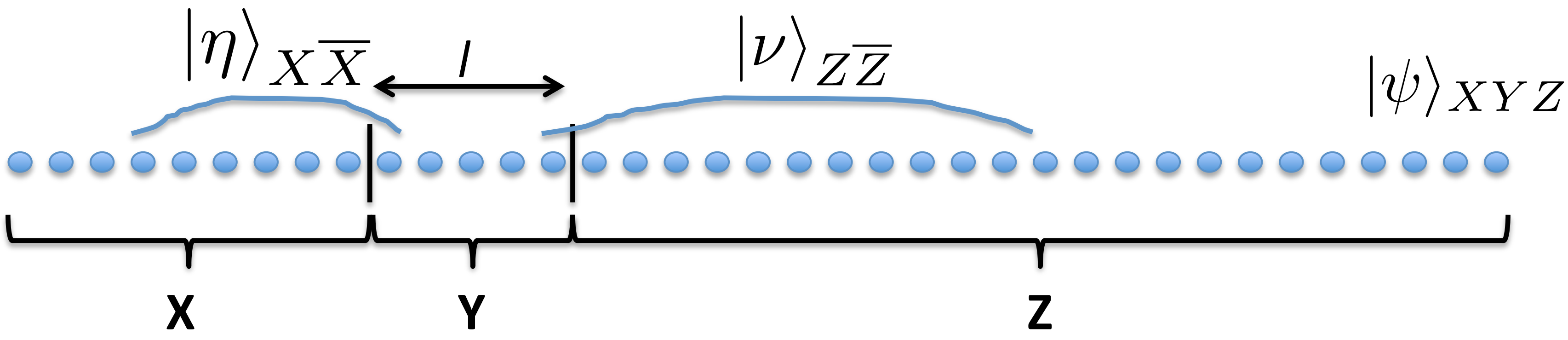
(e.g. QECC, Topological Order)

Area Law from Correlation Length?



Suppose $\rho_{XZ} = \rho_X \otimes \rho_Z$.

Area Law from Correlation Length?

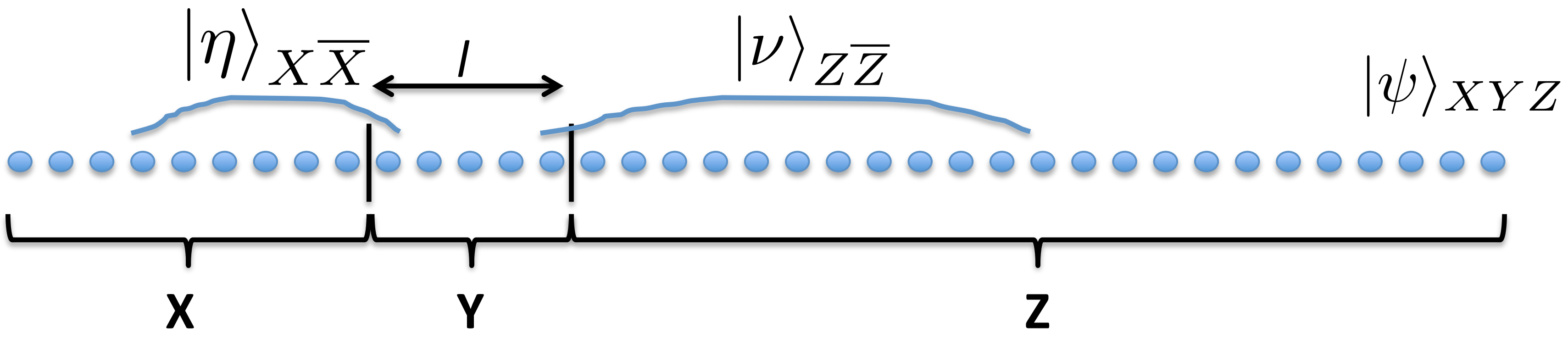


Suppose $\rho_{XZ} = \rho_X \otimes \rho_Z.$

Then $|\psi\rangle_{XYZ} = (U_{\overline{XZ}\to Y} \otimes I_{XZ})|\eta\rangle_{X\overline{X}} \otimes |\nu\rangle_{Z\overline{Z}}$

X is only entangled with Y

Area Law from Correlation Length?



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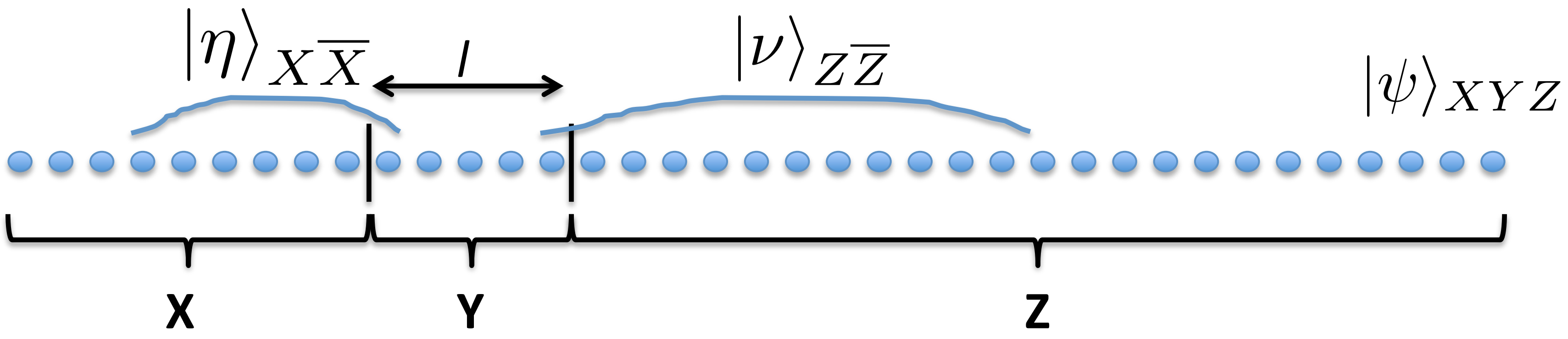
What if merely $\mathrm{cor}(X : Z) \leq 2^{-l/\xi}$?

Area Law from Correlation Length?



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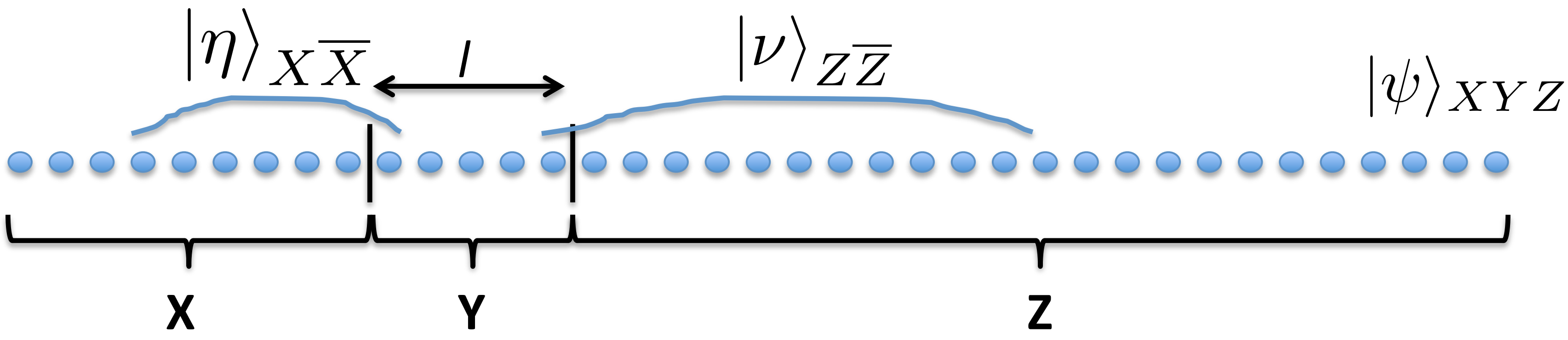

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True (Uhlmann's thm). But we need 1-norm (trace-distance):

$$\rho_{XZ} \approx \rho_X \otimes \rho_Z \iff \|\rho_{XZ} - \rho_X \otimes \rho_Z\|_1 \leq \varepsilon$$

$$\|\rho_{XZ} - \rho_X \otimes \rho_Z\|_1 := 2 \max_{0\leq M\leq I} \mathrm{tr}(M(\rho_{XZ} - \rho_X \otimes \rho_Z))$$

In contrast $\mathrm{cor}(X, Z) := \max_{A,B} \mathrm{tr}((A \otimes B)(\rho_{XZ} - \rho_X \otimes \rho_Z))$

Data Hiding States

Well distinguishable globally, but poorly distinguishable locally

(DiVincenzo, Hayden, Leung, Terhal ’02)

Ex. 1 Antisymmetric Werner state $\omega_{AB} = (I - F)/(d^2 - d)$

$$\mathrm{cor}(A:B) \leq 1/d \qquad \qquad \left\| \omega_{AB} - \omega_A \otimes \omega_B \right\|_1 \approx 1/2$$

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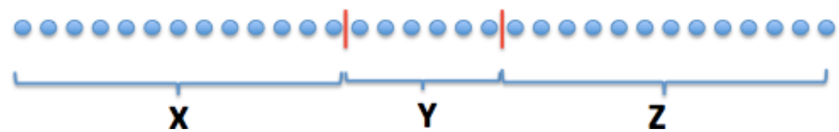
Ex. 2 Random state $|\psi\rangle_{XYZ}$ with $|X|=|Z|$ and $|Y|=l$

$$\mathrm{cor}(X : Z) \leq 2^{-cl}$$

$$S(X) \approx (n-l)/2$$



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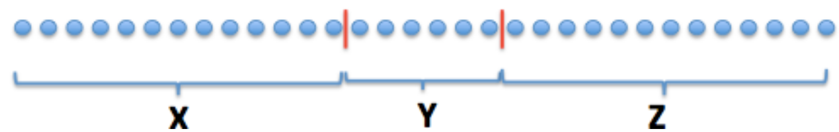
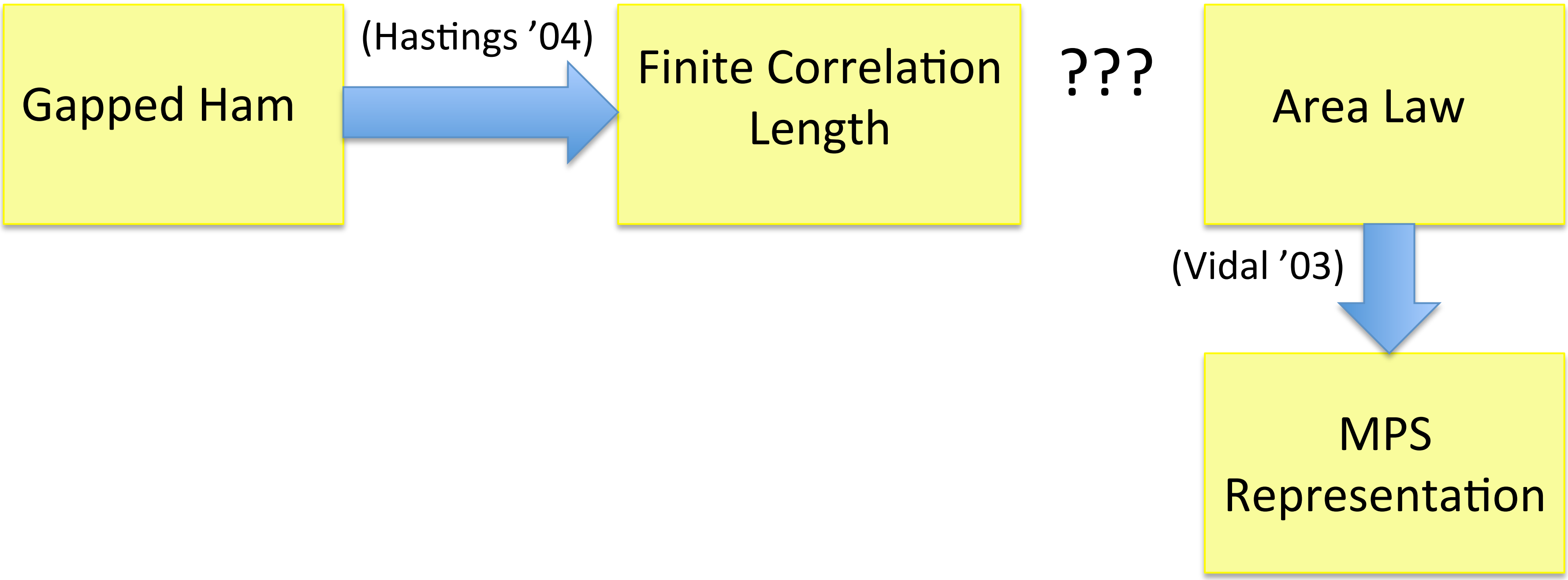


Ex. 3 (Hastings ’07) Quantum Expanders States: States with big entropy but s.t. for every regions X, Z *far away* from edge

$$\mathrm{cor}(X : Z) \leq 2^{-c\,\mathrm{dist}(X,Z)}$$

Area Law in 1D?



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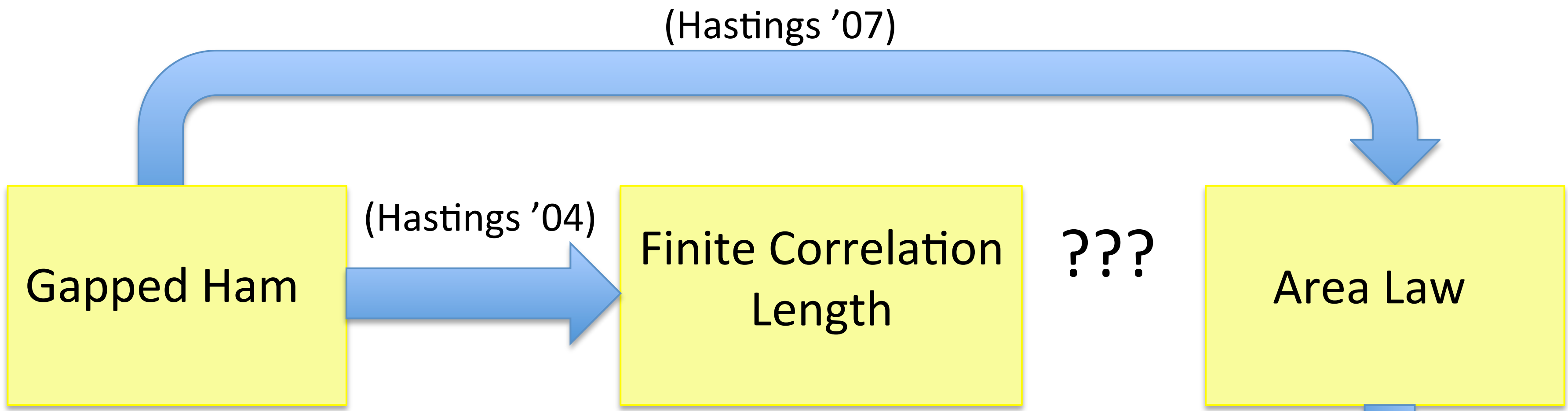
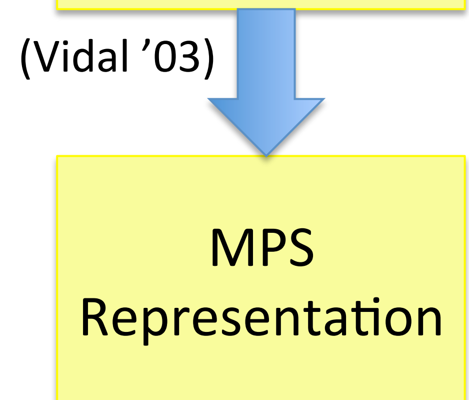


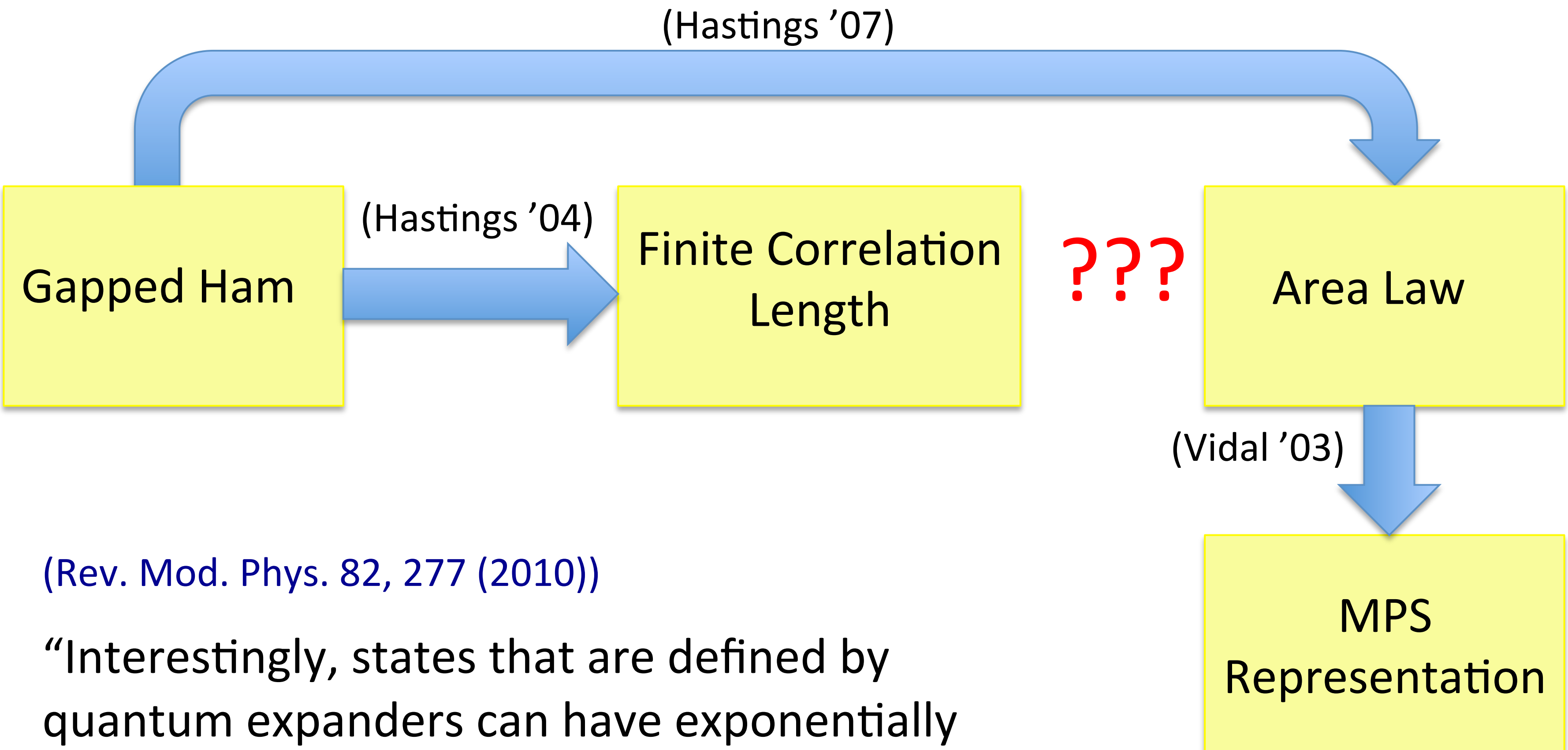


thm (Hastings '07) For H with spectral gap Δ and *unique* groundstate Ψ_0, for every region X,

$$S(X)_\psi \leq \exp(c / \Delta)$$

(Arad, Kitaev, Landau, Vazirani '12) $S(X)_\psi \leq c / \Delta$

Area Law in 1D



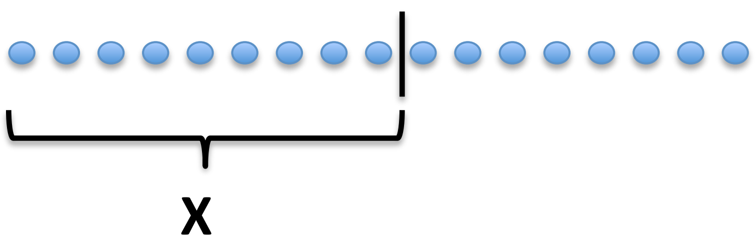
(Rev. Mod. Phys. 82, 277 (2010))

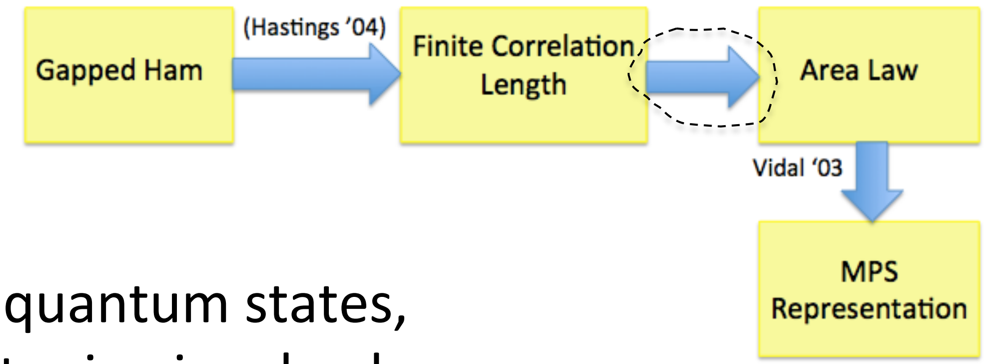
"Interestingly, states that are defined by quantum expanders can have exponentially decaying correlations and still have large entanglement, as has been proven in (…)"

Correlation Length vs Entanglement

thm 1 (B., Horodecki '12) Let $|\psi\rangle_{1,\ldots,n}$ be a quantum state in 1D with correlation length ξ. Then for every X,

$$S(X)_\psi \leq 2^{c\xi}$$





- The statement is only about quantum states, no Hamiltonian involved.
- Applies to *degenerate* groundstates, and *gapless* models with finite correlation length (e.g. systems with mobility gap; many-body localization)

Summing Up

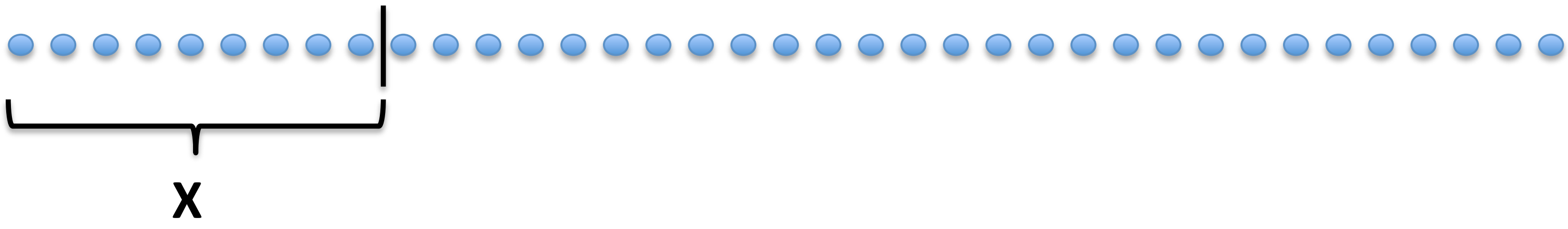
Area law always holds in **1D** whenever there is a **finite correlation length**:

- Groundstates (unique or degenerate) of gapped models
- Eigenstates of models with mobility gap (many-body localization)
- Thermal states at any non-zero temperature
- Steady-state of gapped dissipative dynamics

Implies that in all such cases the state has an **efficient classical parametrization** as a MPS

(Useful for numerics – e.g. DMRG.
Limitations for quantum information processing
e.g. no-go for adiabatic quantum computing in 1D)

Proof Idea

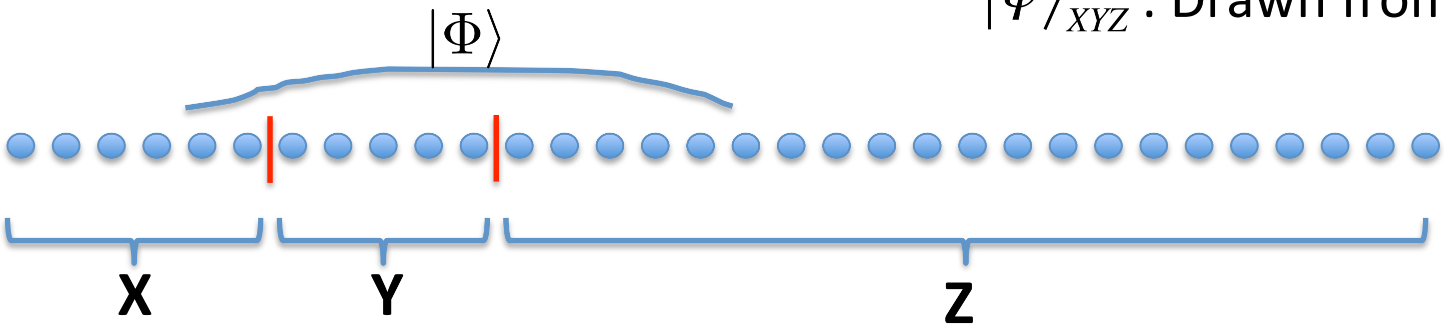


We want to bound the entropy of X using the fact the correlation length of the state is finite.

Need to **relate entropy to correlations**.

Random States Have Big Correl.

$|\psi\rangle_{XYZ}$: Drawn from Haar measure



Let size(XY) < size(Z). W.h.p. $\left\|\rho_{XY} - \tau_X \otimes \tau_Y\right\|_1 \leq 2^{-\Omega(n)}, \quad \tau_X := \mathrm{id}/d_X$

X is **decoupled** from *Y*.

Extensive entropy, but also *large* correlations:

$$U_{Z\to Z_1 Z_2}|\psi\rangle_{XYZ} \approx |\Phi\rangle_{XZ_1} \otimes |\Phi\rangle_{YZ_2}$$

$|\Phi\rangle_{XZ_1}$: Maximally entangled state between XZ_1.

$\mathrm{Cor}(X{:}Z) \geq \mathrm{Cor}(X{:}Z_1) = 1/4 >> 2^{-\Omega(n)}$: **long-range correlations**

Entanglement Distillation

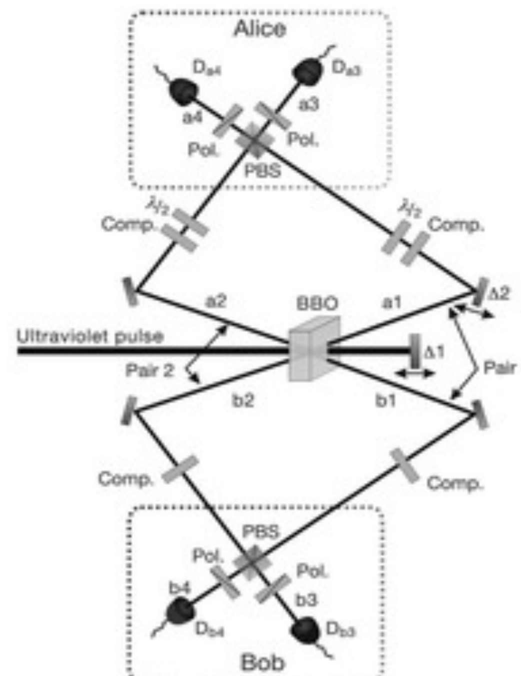
Consists of extracting EPR pairs from bipartite entangled states by **Local Operations and Classical Communication (LOCC)**

Central task in quantum information processing for distributing entanglement over large distances (e.g. entanglement repeater)

ρ_{AB}

LOCC

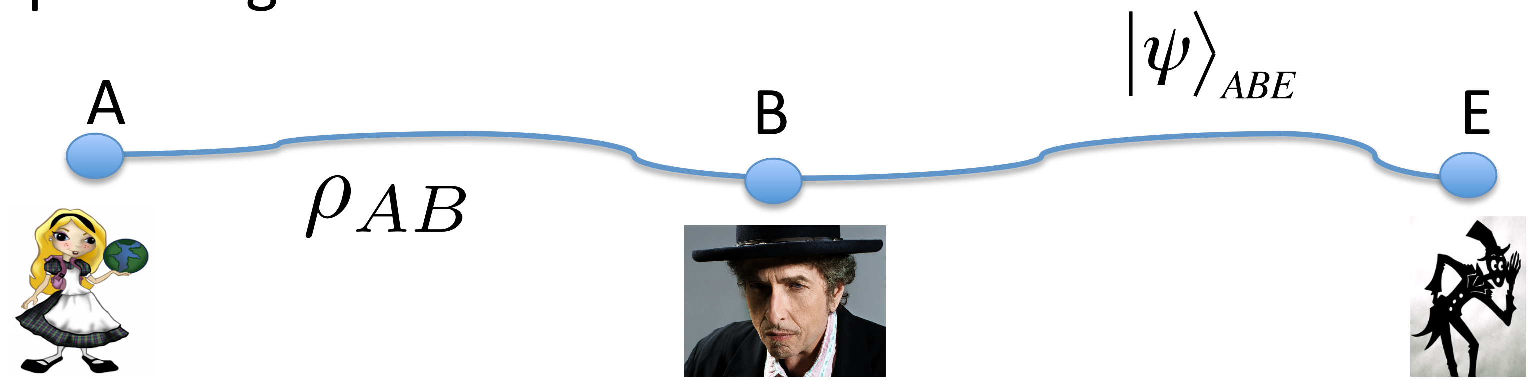
$|\phi^+\rangle\langle\phi^+|^{\otimes m}$



(Pan *et al* ’03)

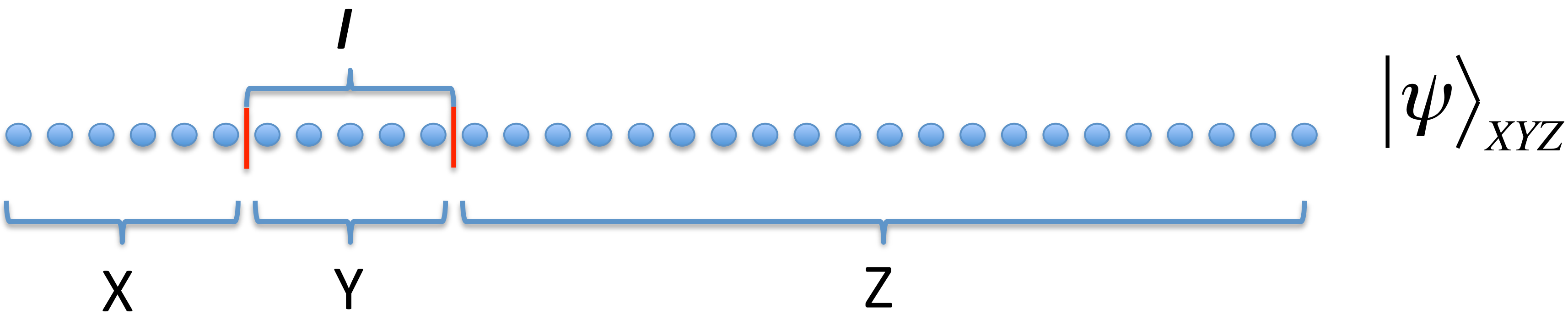
Entanglement Distillation Protocol

We apply entanglement distillation to show large entropy implies large correlations



Entanglement distillation: Given $|\psi\rangle_{ABE}$ Alice can distill -S(A|B) = S(B) – S(AB) EPR pairs with Bob by making a measurement with N≈ $2^{I(A:E)}$ elements, with I(A:E) := S(A) + S(E) – S(AE), and communicating the outcome to Bob. (Devetak, Winter '04)

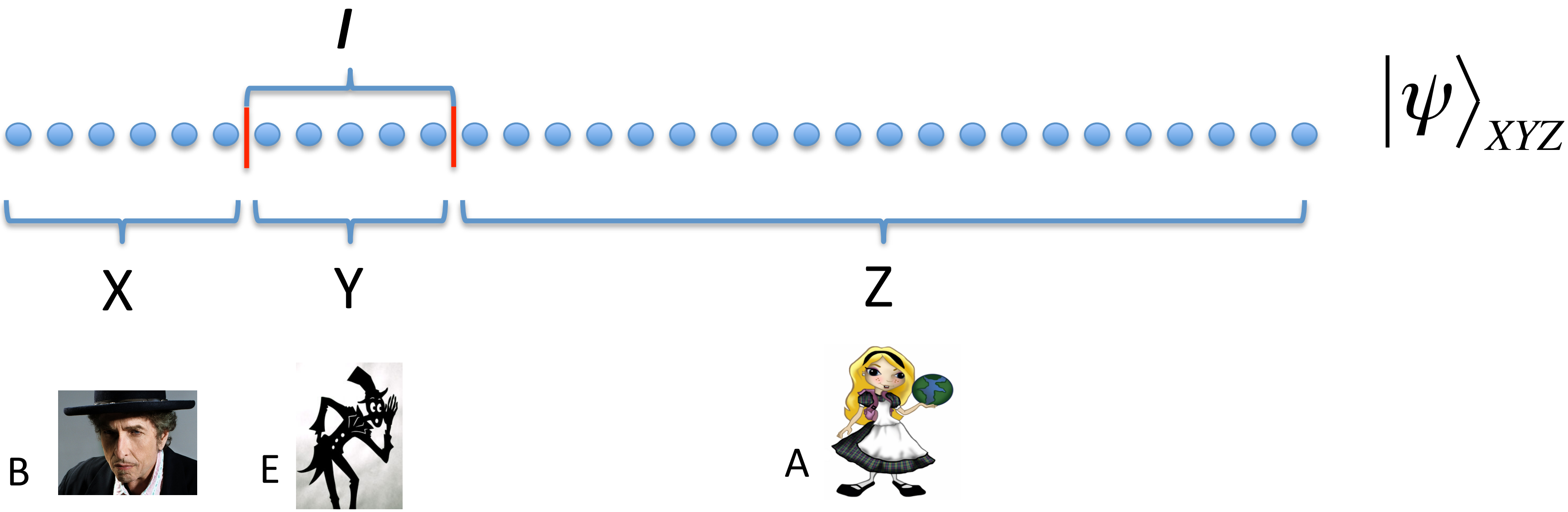
Distillation Bound



B  E  A 

$$S(X) > S(Y) \Rightarrow \quad Cor(X:Z) \geq O\left(2^{-I(X:Y)}\right)$$

Distillation Bound



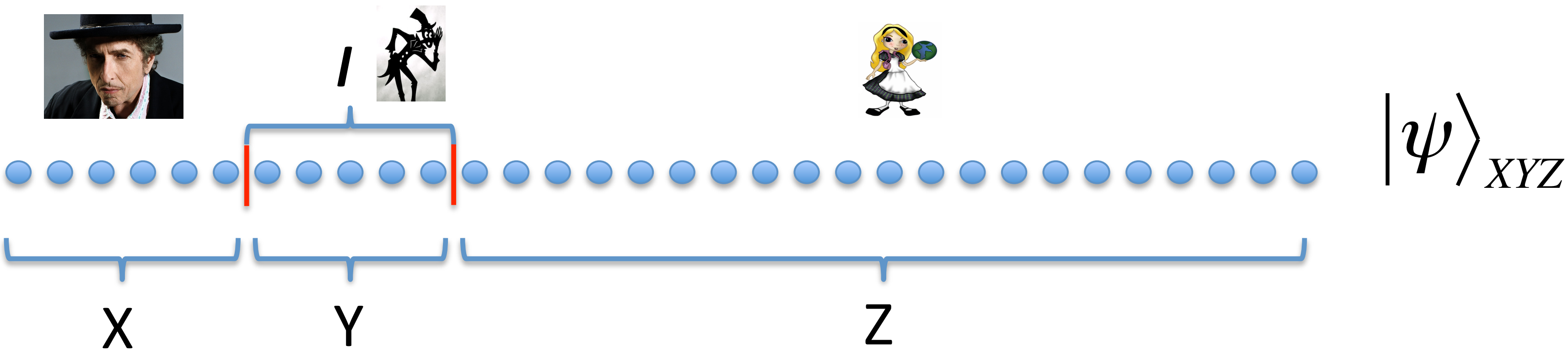
$$S(X) > S(Y) \Rightarrow \quad Cor(X:Z) \geq O\left(2^{-I(X:Y)}\right)$$

$S(X) - S(XZ) > 0$

(EPR pair distillation rate)

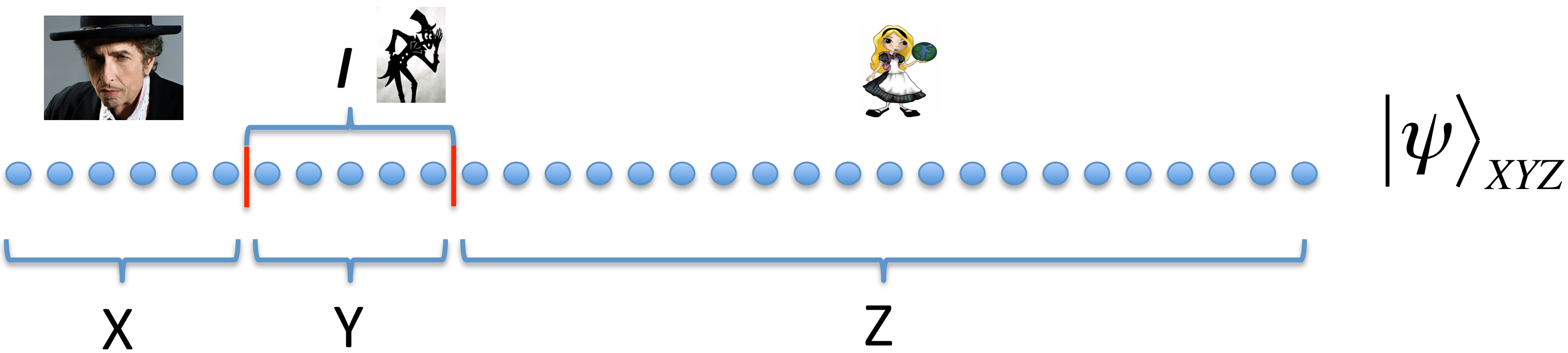
Prob. of getting one of the $2^{I(X:Y)}$ outcomes

Area Law from "Subvolume Law"



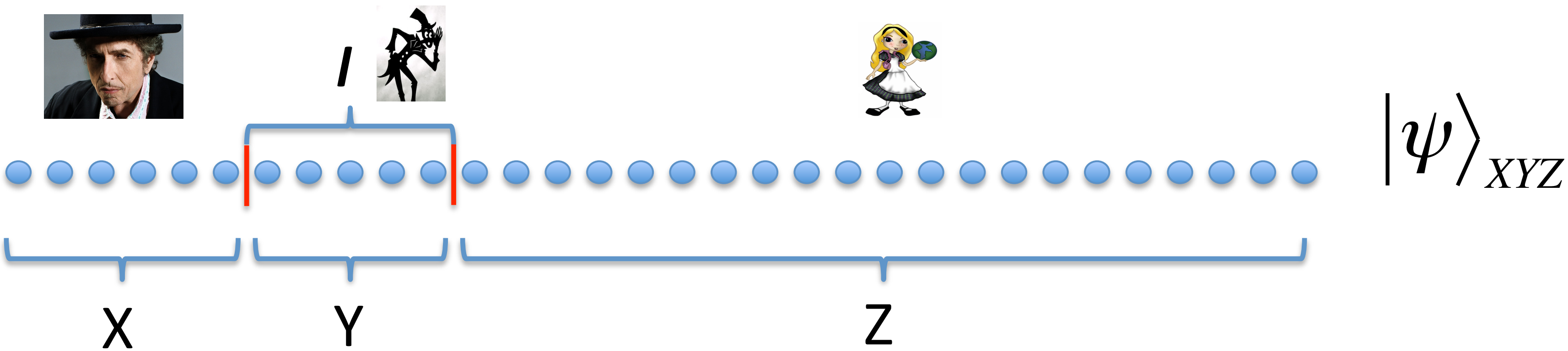
$$S(X) > S(Y) \Rightarrow \quad Cor(X:Z) \geq O\left(2^{-I(X:Y)}\right)$$

Area Law from “Subvolume Law”



$$S(X) \leq S(Y) \;\Leftarrow\; Cor(X:Z) < O\left(2^{-I(X:Y)}\right)$$

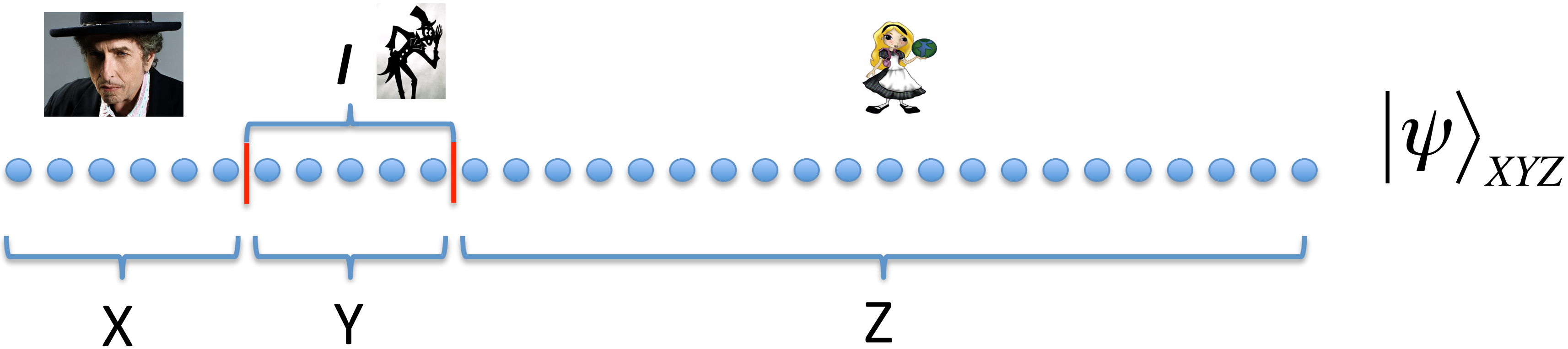
Area Law from "Subvolume Law"



$$S(X) \leq S(Y) \Leftarrow \quad Cor(X:Z) < O\left(2^{-I(X:Y)}\right)$$

Suppose S(Y) < l/(4ξ) ("subvolume law" assumption)

Area Law from "Subvolume Law"



$$S(X) \leq S(Y) \;\Leftarrow\; Cor(X:Z) < O\left(2^{-I(X:Y)}\right)$$

Suppose S(Y) < l/(4ξ) ("subvolume law" assumption)

Since I(X:Y) < 2S(Y) < l/(2ξ), a correlation length ξ implies

$Cor(X:Z) < 2^{-l/\xi} < 2^{-I(X:Y)}$

Thus: S(X) < S(Y)

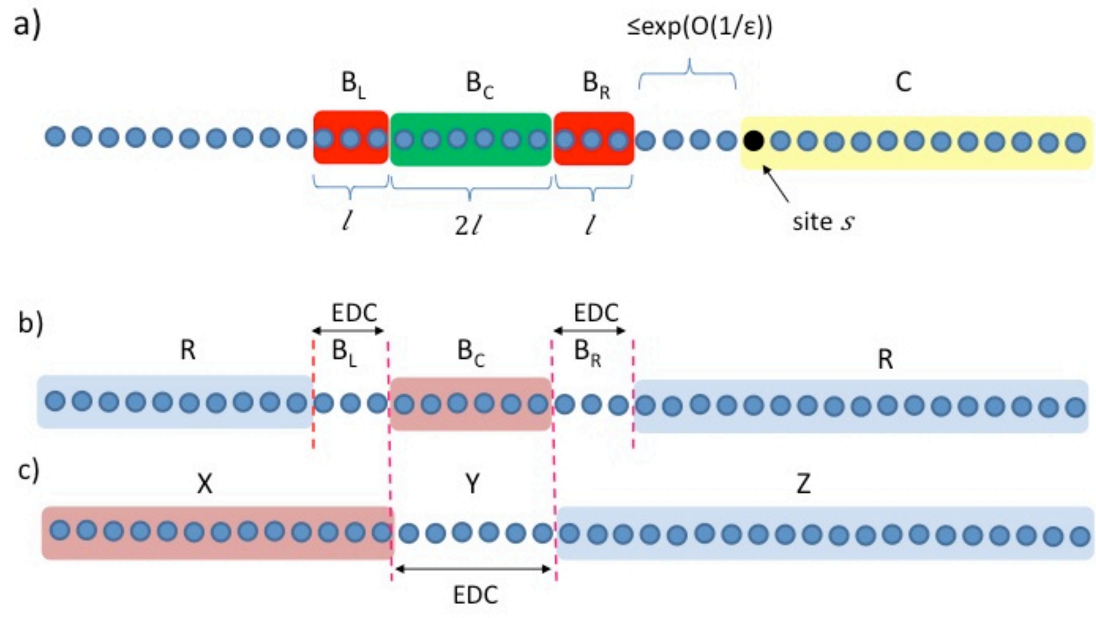
Actual Proof

We apply the bound from entanglement distillation to prove **finite correlation length -> Area Law** in 3 steps:

c. Get area law from finite correlation length under assumption there is a region with "subvolume law"

b. Get region with "subvolume law" from finite corr. length and assumption there is a region of "small mutual information"

a. Show there is always a region of "small mutual info"



Each step uses the assumption of finite correlation length.

Obs: Must use single-shot info theory (Renner *et al*)

Area Law in Higher Dim?

Wide open…

Known proofs in 1D (for groundstates gapped models):

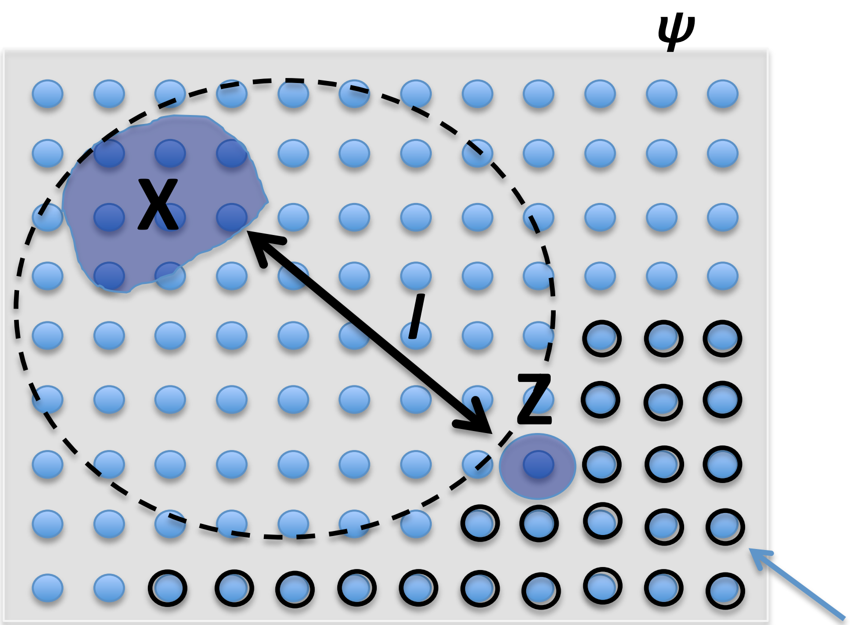
1. **Hastings '07**. Analytical
   (Lieb-Robinson bounds, Fourier analysis,…)
2. **Arad, Kitaev, Landau, Vazirani '13**. Combinatorial
   (Chebyshev polynomial, …)
3. **B., Horodecki '12** (this talk). Information-Theoretical

All fail in higher dimensions….

Area Law in Higher Dim?

New Approach:

"Conditional Correlation length": $\mathbb{E}_{i_1,\ldots,i_k} \operatorname{cor}(X : Z)_{\psi_{i_1,\ldots,i_k}} \leq 2^{-l/\xi}$



$\psi_{i_1,\ldots,i_k}$: post-measured state after measurement on sites (a_1,..., a_k) with outcomes (i_1, ..., i_k) in $\{0, 1\}^k$

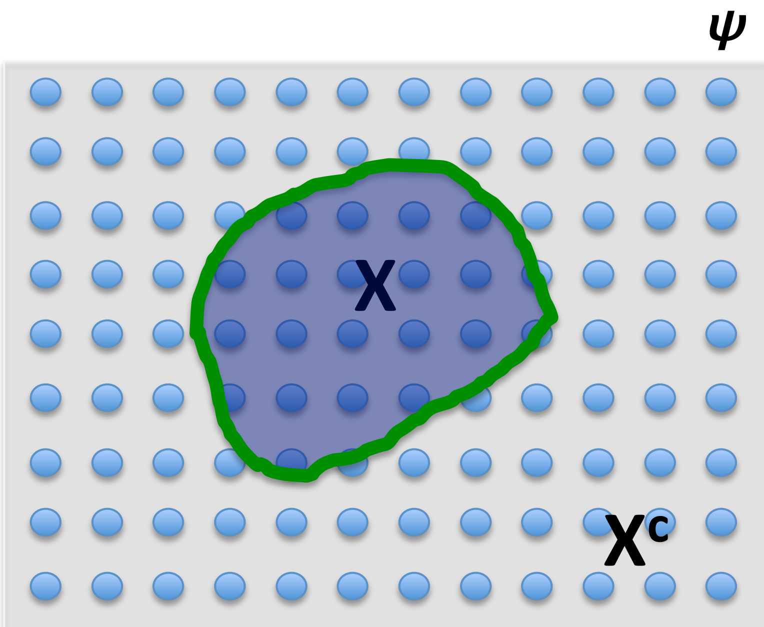
$$|\psi_{i_1,\ldots,i_k}\rangle := \frac{1}{\|\ldots\|} \left(\mathrm{id} \otimes |i_k\rangle\langle i_k|_{a_k} \otimes \ldots \otimes |i_1\rangle\langle i_1|_{a_1}\right) |\psi\rangle$$

Measurement on site a_k

Area Law from Finite *Conditional* Correlation length

thm (B. '14) In any dim, if ψ has conditional correlation length ξ, then

$$S(X)_{\psi} \leq 4\xi \, \text{Area}(X)$$



Which states have a finite conditional correlation length?

Conjecture1: Any groundstate of gapped local Hamiltonian.

Conjecture 2: Any state with a finite correlation length.

Obs: Can prove it for 1D models
(finite CL -> area law -> MPS -> finite CCL)

Area Law from Finite *Conditional* Correlation length

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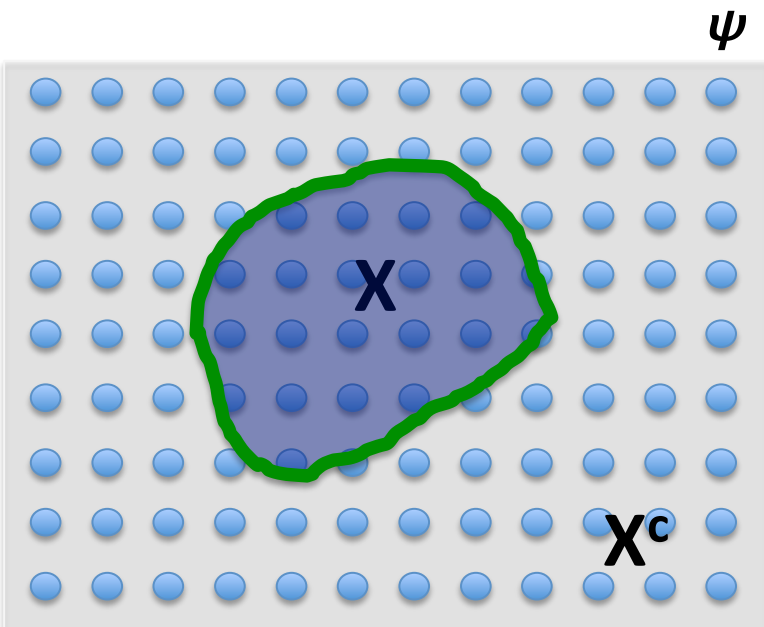

Proof by quantum information theory:

$$\begin{aligned} 2S(X)_\psi &= I(X:X^c)_\psi \\ &\leq I(X:X^c)_\rho + 2\xi \text{Area}(X) \end{aligned}$$

conditional corr. length

$$\rho = \text{id}_X \otimes \Lambda^{\otimes \text{size}(X^c)}(|\psi\rangle\langle\psi|)$$

$$\Lambda(X) = \langle 0|X|0\rangle|0\rangle\langle 0| + \langle 1|X|1\rangle|1\rangle\langle 1|$$

Application to Systems with *Robust* Gap

thm (B. '14) In any dim, if ψ has conditional correlation length ξ, then

$$S(X)_\psi \leq 4\xi \,\text{Area}(X)$$

(Verstraete '14) Groundstates of Hamiltonians with local topological order have finite conditional correlation length.

LTQO : Closely related to "robust gap", i.e. $H + \varepsilon \sum_k V_k$ is gapped for ε small enough and all V_k.

cor Every groundstate of a system with local topological order fulfills area law

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Improves on (Michalakis, Pytel '11) who proved S(X) ≤ Area(X)log(vol(X)).

Obs: *Strict* area law is important, as it allows us to define the concept of topological entanglement entropy (Kitaev, Preskill '05, Levin, Wen '05)

Summary

- Finite correlation length gives an area law for entanglement in 1D. We don't know what happens in higher dimensions.
- More generally, thinking about entanglement from the perspective of quantum information theory is useful.
- Growing body of connections between concepts/techniques in quantum information science and other areas of physics.

Thanks!